Classical and Quantum Radiation Reaction

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Abstract: This thesis reports on work undertaken in comparing the effects of the phenomenon of radiation reaction in classical and quantum theories of electrodynamics. Specifically, it is concerned with the prediction of the change in position of a particle due to the inclusion of the self-force in the theory. We calculate this position shift for the classical theory, treating radiation reaction as a perturbation in line with the reduction of order procedure. We calculate the contributions to the position shift in the $\hbar \to 0$ limit of quantum field theory to order e^2 in the coupling, the order of the classical self-force. These calculations contain the emission and forward scattering one loop processes of quantum electrodynamics. The quantum calculations are completed for the case of a particle represented by a scalar field wave packet and then for a particle represented by the Dirac spinor field. We additionally give an alternative derivation of the scalar results using the interpretation of radiation reaction via a Green's function decomposition, in order to explain and contrast the results achieved.

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Author's declaration

I declare that the work contained in the thesis is original. Chapter 1, Chapter 2 sections 1 and 2 and Appendices B and C are reviews. Chapter 2 section 3, Chapters 3, 4, 5, 6 and 7 and Appendix A are my original work done in collaboration with my supervisor, Dr Atsushi Higuchi. The work contained in Chapters 3, and 4 is reported in [**6**] and [**7**]. The work in Chapter 5 is reported in [**8**].

CHAPTER 1

Introduction

In this chapter we introduce the work presented in this thesis. We introduce the background theory of relevance, the work to be presented and define the classical and quantum theoretical models to be used.

1. Overview

The concept that an accelerated particle radiates is one the of most widely known and used phenomena from the theory of classical electrodynamics. It is thus ironic that the theory of the process and the mechanism behind it is in fact one of the least understood and most debatable areas of classical theory. In truth, there is no real consensus over the correct interpretation of the theory, or even exactly which theory is the correct one to interpret. The problems stem from the attempt to describe the effect that the emission of such radiation would have on the particle itself - *radiation reaction*. That radiation is in fact produced by various systems involving the acceleration of charged particles is an observable experimental fact. The phenomenon is one of the most frequently employed in electromagnetism, for example in the production of radio waves from antennas. [1] The radiation itself carries away energy-momentum which must consequently affect the particle's motion via recoil in order to conserve the energy-momentum of the system. Thus radiation reaction alters the equations of motion of a charged particle. This is, of course, fundamental to our understanding as the equations of motion for a system are one of the most basic underpinnings of a theory. Nevertheless, the effect

[1] Curiously, the plural antennae is used for biological appendages, whereas antennas is the use for equipment sending and receiving electromagnetic waves.

is rarely considered (or even taught in undergraduate courses). The usual focus in classical theory is either the study of the fields given the motion of a charged particle, or the motion of a charged particle given some external field(s). The problem of radiation reaction, on the other hand, is one of the effect on the motion of the particle of its *own* field, hence the frequently employed alternative name, *self-force*. The lack of attention to this effect is possibly due to a combination of factors including,

- The effect of radiation reaction is very small for most purposes; sufficiently small to be ignored.
- The unresolved and/or debatable problems alluded to above which prevent the presentation of a consistent theory on a concrete footing.
- Classical electrodynamics is no longer considered to be the fundamental theory, having been superseded by quantum electrodynamics.

The focus of this project is a comparison of the effects of radiation reaction in both the classical and quantum electrodynamics' theories. In this way, we hope to gain further understanding of how radiation reaction is treated within these theories and how this treatment differs. The fundamental nature of the effects of adding radiation reaction to a model, as one must do to obtain a realistic model, means that a fuller understanding of the nature of radiation reaction is essential. Indeed, it is not only in the theory of electrodynamics that we are presented with this problem and much current research is, at the time of writing, focused on radiation reaction problems in classical gravity.[2]

In the coming sections of this introductory chapter we present a description of the background theory of relevance to the study of radiation reaction and the origin of the work presented here. This work is based on the calculation of the 'position shift', the change in position due to radiation reaction, as a measure of the effect of radiation reaction and from section 6 we then explain the models to be used, detailing the choice of calculations to be performed. We

[2]We shall return to this subject briefly later (Sec.3.3).

introduce the classical model and the conventions and definitions for the quantum field theoretic models on which we shall base our investigations. The body of work that forms this thesis is then split into relevant chapters as follows: In Chapter 2, we introduce and calculate the semiclassical approximations for use in describing the quantum fields in our calculations and in Chapter 3 we calculate the position shift in the classical theory of electrodynamics. In Chapter 4, we then proceed with the calculations using the quantum scalar electrodynamics, calculating the contributions to the position shift and comparing the $\hbar \to 0$ limit with the results from the classical theory. Chapter 5 then gives an alternative derivation for some of the results for the scalar field by using the Green's function decomposition description of radiation reaction in order to gain further understanding and interpret the previously obtained results. Chapter 6 then repeats our calculations for the canonical quantum electrodynamics model based on the Dirac spinor field. The appendices include definitions and calculations which are used and referred to within the main body of the text.

2. Radiation from moving charges

Before considerations are made of our theory of radiation reaction, it would be timely to remind ourselves of some of the basic theory concerning radiation from accelerated charges in flat spacetime.[3] We recall that in the absence of incoming fields, we may write the 4-vector electromagnetic potential A generated by the motion of a charged particle in terms of the retarded Green's function G_- and the particle's 4-current j:

$$A^\mu(x) = \int d^4x' G_-{}^\mu{}_\nu(x - x') j^\nu(x') , \tag{1}$$

with $G_-{}^\mu{}_\nu = \delta^\mu_\nu G_-$ and where our metric signature is represented by $g_{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1)$. The units are chosen so that $c = 1$, where c is the

[3]This explanation is intended as a reminder for those familiar with the theories quoted. For a more in depth discussion [**21**] is a good place to start.

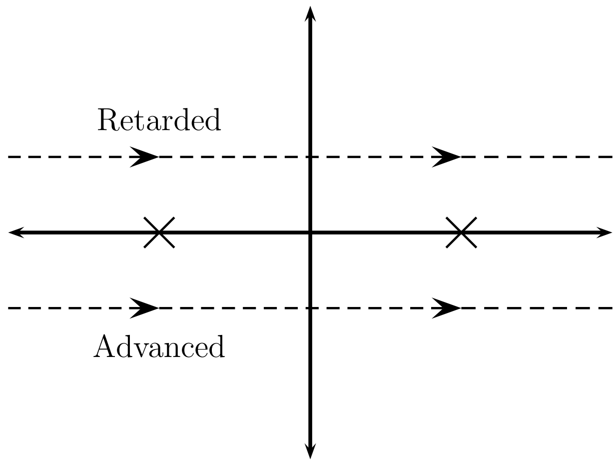

FIGURE 1.1. The contours used by the Retarded and Advanced Green's functions avoiding the poles (X) in the k_0 integration.

speed of light. We also let the electromagnetic field satisfy the Lorentz gauge condition $\partial_\alpha A^\alpha = 0$. The current is given by

$$j^\nu(x) = e \int d\tau \frac{dx^\nu}{d\tau} \delta^4(x - X(\tau)) \,, \tag{2}$$

where $X(\tau)$ is the space-time trajectory of the particle. The point particle nature of the theory is represented by the delta function point distribution. This is technically the retarded potential, with the advanced potential being analogously generated from the advanced Green's function. We canonically choose the retarded solution due to our wish to look at propagation forward in time, which can be seen more explicitly below.

For the electromagnetic field, the Green's functions are the fundamental solutions to the wave equation

$$\Box G(x, x') = \delta^4(x - x') \,, \tag{3}$$

where the translation invariance means that the solutions depend only on $x - x'$, hence $G(x, x') = G(x - x')$. By utilizing the resultant algebraic equation for the Fourier transform, the solutions can be written in the integral form

$$G(x - x') = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-x')}}{k^2} \,. \tag{4}$$

The singularities from the cone $k^2 = 0$ are dealt with via deforming the

contour of integration. The different Green's functions, resulting from the alternative boundary conditions applied to the wave equation, generate the different contours.[4] The retarded/advanced Green's functions are generated using the contours that travel above/below the poles of the k_0 integration viz

$$G_{\mp}(x-x') = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-x')}}{(k_0 \pm i\varepsilon)^2 - \mathbf{k}^2}\,, \tag{5}$$

where the limit $\varepsilon \to 0^+$ is assumed (see Fig. 1.1). This gives

$$G_{\mp}(x-x') = \frac{1}{2\pi}\theta(\pm(t-t'))\delta((x-x')^2)\,. \tag{6}$$

where θ is the Heaviside step function. This expression shows clearly that the support for the retarded and advanced Green's functions lies on the future and past light cones respectively of the particle at x' [5], as is expected from causality reasoning. Fig. 1.2 shows these light cones for x' on a particle trajectory given by $\gamma = X(\tau)$. The potential at x is generated by the point on the world line of the trajectory that intersects x's *past* light cone, which is known as the retarded point. The proper time of the particle at this intersection is known as the retarded time, which we label τ_- here. Likewise for the advanced potential, we have the intersection with x's *future* light cone, at the advanced time τ_+. This is represented by Fig. 1.3.

Returning to our electromagnetic potential generated by the moving charged particle, we have

$$A^\mu(x) = \frac{e}{2\pi}\int d\tau \theta\left(x^0 - X^0(\tau)\right)\delta\left((x-X(\tau))^2\right)\frac{dX^\mu}{d\tau}\,, \tag{7}$$

Labelling $X(\tau_\pm) = X_\pm$ as the advanced/retarded points on the particle trajectory for x, and $\dot{X}_\pm = dX/d\tau(\tau_\pm)$ as the world-velocities at those points, we can solve the integral and write

$$A^\mu(x) = \frac{e}{4\pi}\frac{\dot{X}^\mu_-}{\dot{X}^\nu_-\left(x-X_-\right)_\nu}\,. \tag{8}$$

[4]The difference between the choices must, in the end, correspond to a solution to the homogeneous equation.

[5]That is, x lies on the future/past light cone of x' and consequently, x' lies on the past/future light cone of x.

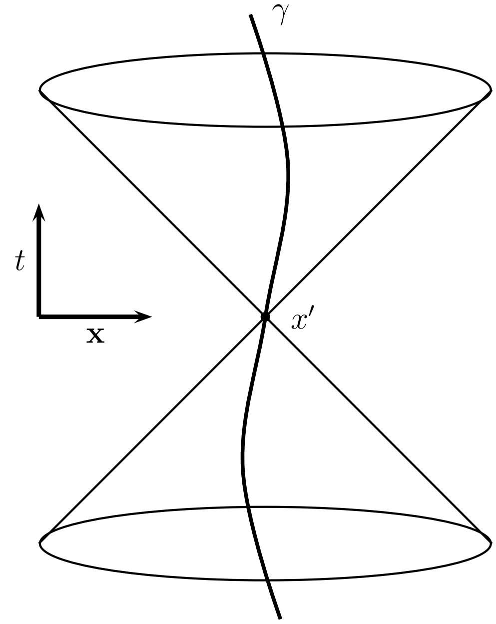

FIGURE 1.2. Light cones of a point on a world line γ.

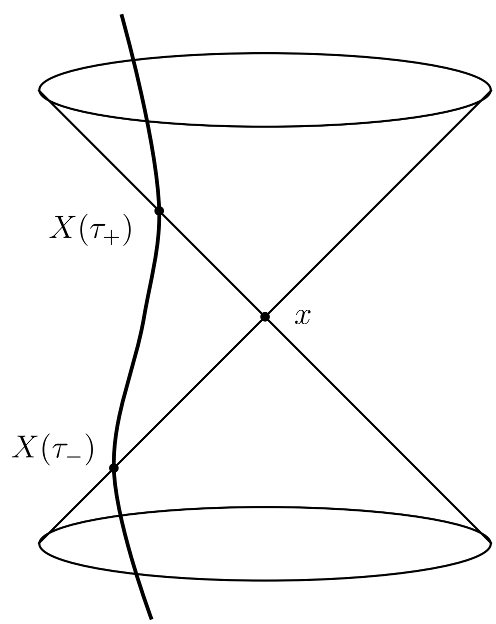

FIGURE 1.3. The intersection of the light cone of point x with the world line of the particle, at the retarded and advanced points.

These are commonly known as the Liénard-Wiechert potentials and are the usual starting place in textbooks and electrodynamics courses for the analysis of the radiation of a moving charge. This expression is fully covariant and can be considered to be the relativistic generalisation of the Coulomb potential.[6] The most common exposition of the potentials and radiation involve a non-covariant form written in terms of the quantity $\mathbf{r} = \mathbf{x} - \mathbf{X}_-$, which we name the radial vector and where due to the null separation of the points the magnitude can be given similarly by $r = |\mathbf{r}| = x^0 - X^0_-$. Hence,

$$A^0 = \frac{e}{4\pi} \frac{1}{(r - \mathbf{r} \cdot \mathbf{v}_-)}, \tag{9}$$

$$\mathbf{A} = \frac{e}{4\pi} \frac{\mathbf{v}_-}{(r - \mathbf{r} \cdot \mathbf{v}_-)}, \tag{10}$$

where $\mathbf{v}_- = d\mathbf{X}/dt(\tau_-)$. The magnetic and electric fields from this potential can then be given by

$$\mathbf{B} = \frac{\mathbf{r} \times \mathbf{E}}{r}, \tag{11}$$

$$\mathbf{E} = \frac{e}{4\pi} \frac{(\mathbf{r} - r\mathbf{v}_-)(1 - \mathbf{v}_-^2) + \mathbf{r} \times [(\mathbf{r} - r\mathbf{v}_-) \times \dot{\mathbf{v}}_-(t)]}{(r - \mathbf{r} \cdot \mathbf{v}_-)^3}, \tag{12}$$

with $\dot{\mathbf{v}}_-(t) = d\mathbf{v}_-/dt$. It is worth recalling from the definitions that all terms on the right hand side are evaluated at the retarded time τ_-. The expressions in (11) and (12) can each be separated into two terms, representing the so called 'velocity fields', which do not depend on the acceleration, and the 'acceleration fields', which do. Introducing the notation $\mathbf{n} = \mathbf{r}/r$, we rewrite the $\mathbf{E}$ field for example, and obtain

$$\mathbf{E} = \frac{e}{4\pi} \left[\frac{(\mathbf{n} - \mathbf{v}_-)(1 - \mathbf{v}_-^2)}{(1 - \mathbf{n} \cdot \mathbf{v}_-) r^2} + \frac{\mathbf{n} \times [(\mathbf{n} - \mathbf{v}_-) \times \dot{\mathbf{v}}_-]}{(1 - \mathbf{n} \cdot \mathbf{v}_-) r} \right]. \tag{13}$$

The first term, the velocity field, can be seen to be an inverse square field, thus effectively a static Coulomb type field. The second term is the acceleration field, which we see has the inverse radial dependence one would expect from a radiation field. For this field we can also confirm that both $\mathbf{E}$ and $\mathbf{B}$ are transverse to the radial vector. That our interpretation of the velocity field as

[6] In the frame in which $\dot{X}_- = (1, 0, 0, 0)$ we obtain the Coulomb potential.

a Coulomb type field is valid can be confirmed by consideration of the situation in which the particle undergoes uniform motion, hence when the acceleration field naturally vanishes. The remaining contribution should be the Lorentz transformation of the static Coulomb field. This can indeed be shown to be the case. Therefore, this transformation to or from the static frame implies that the velocity field should not affect the motion of the particle[7] nor consequently cause any modifications to the equations of motion. We thus arrived at the conclusion that the acceleration of a charged particle induces a radiation field, i.e. the emission of radiation from the particle.

Having predicted that an accelerated charged particle will radiate, one is naturally inclined to ask 'how much radiation is expected?' Now the energy flux $\mathcal{E}$ across a sphere of finite size and centered on the particle is given in terms of the Poynting vector, viz

$$\begin{aligned} \frac{d\mathcal{E}}{dt} &= \int_S d\mathbf{S} \cdot (\mathbf{E} \times \mathbf{B}) \\ &= \int_S d\Omega (\mathbf{r} \times \mathbf{E})^2 \,. \end{aligned} \tag{14}$$

The last line demonstrates that the energy flux is positive. For the non-relativistic case, i.e. at small velocities, the electric acceleration field contribution is the standard dipole field

$$\mathbf{E}_a = \frac{e}{4\pi r} \mathbf{n} \times (\mathbf{n} \times \dot{\mathbf{v}}_-) \ . \tag{15}$$

The radiated power[8] P in this limit is known as the Larmor formula:

$$P = \frac{d\mathcal{E}}{dt} = \frac{2}{3}\frac{e^2}{4\pi}\dot{\mathbf{v}}^2 \,. \tag{16}$$

The relativistic generalization, written in terms of the energy-momentum 4-vector p of the particle, is given by

$$P = -\frac{2}{3}\frac{e^2}{4\pi m^2}\left(\frac{dp^\mu}{d\tau}\frac{dp_\mu}{d\tau}\right) \ . \tag{17}$$

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[7]Indeed, one does not expect a static particle to move due to its own Coulomb field.

[8]Given that we are now considering the situation at the particle itself, the time in question is in fact the retarded time, and the 'retarded' subscript is henceforth redundant.

3. Radiation Reaction

So far we have looked only at those areas usually considered, namely the effect of the interaction with external fields on a particle (the acceleration of our moving charge) and more chiefly above, the fields produced by such a moving charge and the consequent radiation. What we have yet to consider is of course the effect of this radiation on the motion of the charge. That is, what are the effects on the motion of the particle of its own fields? Put another way, what are the self-interaction effects? As previously mentioned, the radiation carries away energy-momentum and thus one expects the particle's energy-momentum to be affected and hence its motion. We have said that this classical radiative correction is frequently neglected, and one of the main reasons given was that the effect is very small. The approximation is justified provided that the energy concerned is small in comparison with the typical energies of the problem under consideration. Let us consider a period of interaction of time T and with typical resultant acceleration a. The energy of the emitted radiation $\mathcal{E}_{\mathrm{r}}$ is, from above, of order

$$\mathcal{E}_{\mathrm{r}} \sim \frac{2}{3}\frac{e^2}{4\pi}a^2 T\,. \tag{18}$$

The change in the particle's energy $\mathcal{E}_{\mathrm{p}}$ is by comparison of order

$$\mathcal{E}_{\mathrm{p}} \sim m a^2 T^2\,. \tag{19}$$

The demand that $\mathcal{E}_{\mathrm{p}} \gg \mathcal{E}_{\mathrm{r}}$ leads to the relation for the interaction time period

$$T \gg \tau_e = \frac{2}{3}\frac{e^2}{4\pi m}\,, \tag{20}$$

where we have defined the characteristic radiation time τ_e. We note that $\tau_e = 2r_e/3$ where r_e is the classical electron radius and consequently, τ_e is of the order of the time taken for light-signals to travel this distance.[9] For time scales in excess of this period, the corrections can be justifiably ignored. It is evident that this set of larger time scales effectively contains all classical phenomena. Indeed, for lower scales, we would expect to have reached to

[9] Recall our units $c = 1$.

limit of validity of the classical theory and expect quantum effects to become important.

3.1. Abraham-Lorentz-Dirac Force. Below we shall present a description of the canonical classical theory of radiation reaction, the alteration to the equations of motion given by Abraham-Lorentz-Dirac force, and describe some of the associated pathologies. A classical point particle moving under the influence of some external (non-zero) potential, producing a force F_{ext}, is accelerated and the equations of motion given by

$$m\frac{d^2x^\mu}{d\tau^2} = F^\mu_{\text{ext}}\,, \tag{21}$$

where x^μ are the space-time coordinates of the particle at proper time τ. A charged particle emits radiation when accelerated in such a potential and as stated above this will affect the motion of the particle and thus the equations of motion. We could consider the correction as the effect of the addition of an extra force F_{R} on the right hand side of (21):

$$m\frac{d^2x^\mu}{d\tau^2} = F^\mu_{\text{ext}} + F^\mu_{\text{R}}\,. \tag{22}$$

We call this additional force the *radiation reaction force*. Considering for a moment the non-relativistic approximation, as described by the Larmor emission power (16), we note that there are certain restrictions on F_{R}. Given that when there is no acceleration, there should be no radiation, and thus no reaction to it, F_{R} should vanish if $\dot{\mathbf{v}} = 0$. In addition, the only parameter available to use is the characteristic time, hence it is likely to feature in the force. In fact, it is likely to feature at first order, given that the power radiated is of order e^2, in common with τ_e, and that furthermore a sign change on the charge cannot alter the result. One approach is to demand that the work done over the period of interaction is the negative of the total energy radiated i.e.

$$\int_T dt\,\mathbf{F}_{\text{R}}\cdot\mathbf{v} = -\int_T dt\,m\tau_e\dot{\mathbf{v}}^2\,. \tag{23}$$

Integrating the right hand side by parts, then given the assumption that either periodic motion or that $\dot{\mathbf{v}}\cdot\mathbf{v} = 0$ at the end points of the interaction period,

the surface term vanishes and the remainder leads to the conclusion

$$\mathbf{F}_{\mathrm{R}} = m\tau_e\ddot{\mathbf{v}} = \frac{2}{3}\frac{e^2}{4\pi}\ddot{\mathbf{v}}\,, \tag{24}$$

where $\ddot{\mathbf{v}} = d^2\mathbf{v}/dt^2 = d^3\mathbf{x}/dt^3$. This reasoning leads to what is commonly referred to as the Abraham-Lorentz equation of motion:

$$m\dot{\mathbf{v}} = \mathbf{F}_{\mathrm{ext}} + m\tau_e\ddot{\mathbf{v}}\,, \tag{25}$$

(See [**1**] and [**2**]). The resultant equation of motion is different to that which one usually encounters in mechanics due to the presence of third-order differential terms. The inclusion of such a term would imply that a third initial condition would be needed in addition to the position and velocity. It is indeed this type of term that is the source of the debate over the physical correctness and interpretation of this theory. The problem remains when we remove our un-physical non-relativistic approximation.

The fully relativistic generalization of the radiation reaction force was first obtained by Dirac, using the local conservation of energy and momentum [**3**]. The Abraham-Lorentz-Dirac force is the canonical model of radiation reaction in classical electrodynamics, commonly referred to as the Lorentz-Dirac force

$$F^{\mu}_{\mathrm{LD}} = \frac{2\alpha_c}{3}\left[\frac{d^3x^{\mu}}{d\tau^3} + \frac{dx^{\mu}}{d\tau}\left(\frac{d^2x^{\nu}}{d\tau^2}\frac{d^2x_{\nu}}{d\tau^2}\right)\right]\,, \tag{26}$$

where we define $\alpha_c \equiv e^2/4\pi$. Due to the orthogonality of the world-velocity and its proper time derivative

$$\frac{dx^{\mu}}{d\tau}\frac{d^2x_{\mu}}{d\tau^2} = 0\,, \tag{27}$$

the Lorentz-Dirac force is often alternatively written

$$F^{\mu}_{\mathrm{R}} = F^{\mu}_{\mathrm{LD}} \equiv \frac{2\alpha_c}{3}\left[\delta^{\mu}_{\nu} - \frac{dx^{\mu}}{d\tau}\frac{dx_{\nu}}{d\tau}\right]\frac{d^3x^{\nu}}{d\tau^3}\,. \tag{28}$$

The non-relativistic (25) is the result in the special Lorentz frame which is momentarily co-moving with the particle. In both cases, we see the presence of a third-derivative term, usually referred to as the Schott term. Not only is this type of differential equation fundamentally different to that which is expected in dynamical motion, it also presents us with problematic un-physical

solutions. Using the non-relativistic case for simplicity, we rearrange as an inhomogeneous differential equation

$$m\left(\dot{v} - \tau_e \ddot{v}\right) = \mathbf{F}_{\text{ext}}\,. \tag{29}$$

Now, in the homogenous case, i.e. in the absence of any external force, the above equation presents us with two possible solutions

$$\dot{v}(t) = \begin{cases} 0 \\ \dot{v}(0)e^{t/\tau_e} \end{cases}\,. \tag{30}$$

The second solution is referred to as a 'runaway' solution. It would involve a particle effectively accelerating under its own radiation reaction and is not physically acceptable, let alone an observed phenomenon. Additionally, the reader may note that it breaks the boundary conditions imposed during the above derivation which caused the annihilation of the surface term. In order to restrict ourselves to physical solutions, we must add these boundary conditions and in particular demand that should $\mathbf{F}_{\text{ext}} \to 0$ as $t \to \infty$ then $\dot{\mathbf{v}}$ should also vanish. With the addition of these conditions, we may produce an integro-differential form of the Lorentz-Dirac equation, free from the troublesome higher-derivative induced runaways:

$$m\dot{\mathbf{v}}(t) = \int_0^\infty dt' e^{-t'} \mathbf{F}_{\text{ext}}(t + \tau_e t')\,. \tag{31}$$

Unfortunately, this version of the equation is plagued by an alternative problem: pre-acceleration. Consider the case in which the external force is 'switched on' at $t = 0$ i.e.

$$\mathbf{F}_{\text{ext}} \begin{cases} = 0 & \text{if } t < 0 \\ \neq 0 & \text{if } t \geq 0 \end{cases}\,. \tag{32}$$

The reader will note that (31) implies that the acceleration of the particle is not zero for $t < 0$, but instead begins at a time of order $-\tau_e$. Hence the particle accelerates *before* the force is applied. This situation is represented in Fig. 1.4. We note that again, the timescale with which we find ourselves concerned is τ_e.

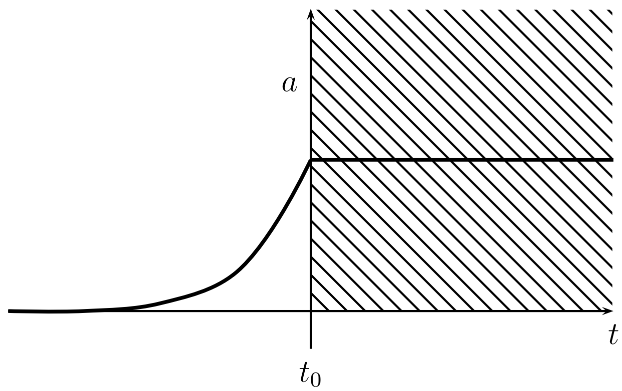

FIGURE 1.4. Pre-acceleration of a charged particle.

Given the previous discussion on the size of the radiative correction, we note that this timescale is beyond the expected validity of classical electrodynamics, and thus the pre-acceleration effect can be considered classically unobservable.

Whilst the 'small' nature of the correction may be a good enough reason to ignore the problems, or indeed the entire phenomenon, for most practical purposes it is hardly satisfying from a fundamental theoretical perspective. If we wish to obtain a proper understanding of the dynamics then we must a) consider radiation reaction and b) attempt to understand and ultimately solve the problems with the current theory. One conclusion we can take from the current situation is that the Lorentz-Dirac theory is at most only approximately accurate. From our discussion, we note that it appears that this accuracy extends only as long as the radiative correction is small. In this regime, one can then treat radiation reaction as a perturbative correction to the motion of the particle. If one proceeds as such, then the problematic third derivative term, treated as a perturbation, can be considered as referring to the acceleration along the original trajectory (rather than the perturbed self-interacting trajectory) viz

$$m\frac{da^\mu}{d\tau} = \frac{dF^\mu_{\text{ext}}}{d\tau}\,. \tag{33}$$

Substituting this relation back into (28), the equations of motion give a second order differential equation,

$$m\frac{d^2x^\mu}{d\tau^2} = F^\mu_{\text{ext}} + \frac{2\alpha_c}{3m}\left[\delta^\mu_\nu - \frac{dx^\mu}{d\tau}\frac{dx_\nu}{d\tau}\right]\frac{dF^\nu_{\text{ext}}}{d\tau}\,. \tag{34}$$

Consequently, the treatment of radiation reaction as a perturbation is akin to a reduction of order process on the differential equations of the motion. With the, dynamically standard, second order equation we are free from run-away and pre-acceleration solutions. In (34), we now have a pathology-free theory for radiation reaction in classical electrodynamics which we can proceed to use, subject to the constraints mentioned. It is in fact this reduced order type of radiation reaction force that is most commonly utilized for calculations involving the self-interaction with electromagnetic and also other fields. The perturbative treatment of the radiative corrections is also familiar as the main calculation tool for quantum electrodynamics (QED). QED is most frequently dealt with using perturbation theory, leading to the ubiquitous Feynman diagrams representing the perturbation expansion terms. This similarity in treatment is one of the motivations for the focus of this work.

The problems with the full theory imply a problem with the starting assumptions. Whilst classical mechanics is of course now known to be based on erroneous grounds[10], due to the need for quantization, we can also query the legitimacy of the point particle model for example, just as extended models are proposed as alternatives in quantum theories. Over the years, a number of different alternative models have been proposed, usually by the addition of either further constraints or changes to the fundamental model. In [**13**], Ford and O'Connell drop the point particle model. Using a particular electron

[10]One ought really to say 'inaccurate grounds' as, along with most modern developments in theory, whilst incorrect, calling the theory false masks the fact that it is remarkably successful in most regimes and also that any new theory must reproduce the results of classical mechanics within their range of validity.

structure type[11] they show that the pathologies in the non-relativistic radiation reaction are due to the point-particle assumption. Under certain restrictions, they reproduce the reduced order equation from this direction.

So far, we have merely stated the Lorentz-Dirac force as Dirac's relativistic generalization to the Abraham-Lorentz force. Dirac derived this equation using considerations of local energy-momentum [**3**]. More explicitly, enclosing the world line of the particle with a 'world-tube', Dirac calculated the energy-momentum flux of the electromagnetic field. Incidently, the shape of the world-tube is irrelevant, provided that the end surfaces are the same. The generalised space-time formulations of Gauss' theorem can be utilized to show that the flux over a deformed tube is the same as the original.[12] By conservation of momentum, the change in the particles momentum can then be deduced as the balance.[13] Following such a line of reasoning will lead one to the Lorentz-Dirac equation. However, to achieve this result the reader will have to make a modification to the physical mass by subtracting the (infinite) contribution from the rate of change of the bound energy-momentum. Thus Dirac found that he had to renormalise the mass by subtracting the infinite contribution of the electro-static self-energy m_{em}, viz

$$ma^{\mu} = \frac{2}{3}\alpha_c \left(\dot{a}^{\mu} - a^2 u^{\mu}\right) \tag{35}$$

[11]Following Feynman, Ford and O'Connell use a form factor $\Omega^2/(\Omega^2+\omega^2)$ dependent on a cut-off frequency Ω and where ω is the typical frequency of the external force.

[12][**18**] demonstrates the equality in this context during the discussion of Dirac's derivation. The derivation in this paper differs slightly from the original, but follows the same basic idea.

[13]Technically, one of Dirac's postulates is that the change in the mechanical energy-momentum of the particle is balanced by that for the electromagnetic field.

where $m = m_0 + m_{\text{em}}$. [14] Lorentz had also had to perform a similar manipulation for the Abraham-Lorentz force. This is another similarity with quantum field theory and in fact many students of theoretical physics are likely to come across the concept of renormalisation in quantum electrodynamics *before* the classical theory. This procedure naturally begs the question concerning the similarities and differences between the classical and quantum renormalisations and thus provides a further motivation for the work presented here.

The separation of the electrostatic contribution from the radiative contribution was noted in the discussion above on the Liénard-Wiechert potentials. Here we mentioned that the electrostatic, short-range contribution was effectively the generalisation of the Coulomb force. Considering the Coulomb force at the (point) particle, one sees why this contribution is infinite. From this perspective, it is also clearer why one would wish this self-energy to be considered part of the left-hand side of (35), as part of the mass, rather than on the right.

3.2. Green's function decomposition. An alternative derivation of the radiation reaction force is motivated by the singular self-energy contribution. We wish to consider the interaction of the particle with its own field. Now, the particle's field can be generated from the retarded Green's function. The reader will recall that the action of the wave operator on the retarded Green's function is to produce a delta function

$$\Box G_{-}(x, x') = \delta^4(x - x')\,, \tag{36}$$

where the distribution is singular at $x = x'$ i.e. at the particle itself. This Green's function was used to generate the electromagnetic Liénard-Wiechert potentials, which in turn were shown to have a singular contribution and a

[14] The expression for the particle's energy-momentum is also not as straight forward as $m_0 a^\mu$, but consideration must be given to the end surfaces of the world-tube. These complications are detailed in [**18**] and in more detail [**4**] and are not as important to the main discussion here.

regular, radiative contribution. This derivation is based on a similar decomposition to the retarded Green's function.

The theory of electromagnetism, contained in Maxwell's equations, is time-symmetric. The process of radiation reaction however is not a time-symmetric process; whilst the emission of radiation from a particle would transform on time reversal to the absorption, the self-energy contributions ought to be the same i.e. time symmetric. We note that in choosing the retarded potential, we broke the time-reversal symmetry of the theory, in order to accommodate our 'time arrow' . Starting from the time-symmetric theory, the opposite choice, of the advanced Green's function, could technically have been made. We thus re-introduce the symmetry by taking the linear combination of Green's functions

$$G_S = \frac{1}{2}\left[G_- + G_+\right] \,, \tag{37}$$

which is a solution to the *inhomogeneous* wave equation. Alternatively, we have the antisymmetric combination

$$G_R = \frac{1}{2}\left[G_- - G_+\right] \,. \tag{38}$$

These two Green's functions form a decomposition of the retarded G_-. [15] Now, as stated above, G_S solves the inhomogeneous wave equation

$$\Box G_S = \delta^4 \,, \tag{39}$$

whilst G_R solves the homogeneous equation

$$\Box G_R = 0 \,. \tag{40}$$

The singular nature of the retarded potential is thus entirely contained within the field generated G_S. With reference to the Liénard-Wiechert potentials, we would consequently hope to assign this contribution as the singular self-energy. Indeed, it can be shown that the singular field

$$A^\mu_S(x) = \int d^4x' G^\mu_{S\nu'}(x,x') j^{\nu'}(x') \,, \tag{41}$$

[15] $G_- = G_R + G_S$.

does *not* affect the motion of the particle (see [**19**]). We thus consider this to indeed be the self-energy Coulomb-like time-symmetric contribution.[16] The remaining contribution G_R, which generates a regular field and is not time-symmetric, we now postulate as the 'radiative' Green's function solution which is responsible for radiation reaction.[17]

The action of the particle's (retarded) field on itself is therefore split into an infinite correction to the mass, generated by G_S, and the remainder $G_- - G_S = G_R$ acts on the particle to produce the radiation reaction force. Explicitly, the radiative field A_R is given by

$$A^\mu_R(x) = \int d^4x' G^\mu_{R\nu'}(x,x') j^{\nu'}(x') \,. \tag{42}$$

The field tensor F^R acting on the particle is then

$$F^R_{\mu\nu} = \nabla_\mu A^R_\nu - \nabla_\nu A^R_\mu \,, \tag{43}$$

and the force is given by the Lorentz force from this field tensor leading to the equations of motion which, with the external field already acting on the particle, are

$$ma_\mu = F^{\text{ext}}_\mu + eF^R_{\mu\nu} u^\nu \,. \tag{44}$$

Using this postulated source for the radiation reaction field, the above equation of motion gives the Lorentz-Dirac force [**18**].

3.3. Curved Space and Gravitational Radiation Reaction. Much recent work on classical radiation reaction has been concentrating on the motion in curved space. Here we briefly mention some interesting extensions to curved space, and to the self-interaction of other fields. In these comments we follow Poisson's excellent review article [**19**] on *Radiation reaction of point particles in curved space*, to which the reader is referred for a detailed pedagogic introduction. The main references for the curved space work in this subsection

[16]The time symmetry means that there should be equal amounts of incoming and outgoing radiation and thus should not affect the motion.

[17]We note here that it is the behaviour at the particle's worldline of the *fields* generated by $G_{R,S}$ that is regular or singular.

use the metric signature $(-+++)$, thus for consistency with these works we shall temporarily adopt this signature for this (and only this) subsection.[18] The extension of the Lorentz-Dirac equation to curved space was originally given by DeWitt and Brehme in 1960 [**30**].[19] The Green's function method for the derivation can be extended to the solutions to the wave equation in curved space. If, as is usually the case, the space-time is globally hyperbolic[20] then there exist unique advanced and retarded solutions to the wave equation. However, these Green's functions have additional features compared to their Minkowski space cousins when considering the support. Recall that in flat space, the support of Green's functions was *on* the light cones. In curved space we have the possibility of interaction between the radiation and the curvature - scattering off the curvature - leading to the propagation of electromagnetic waves at speeds *up to and including* the speed of light. With respect to the Green's functions, this means that the support is extended within the light cones as well as on them. For example, the retarded field, generated from the retarded Green's function, is now dependent on the entire history of the world-line of the particle, up to and including the retarded point. Similarly, the advanced field is dependent on the entire future of the world line, after and including the advanced point. As in the flat space calculations, we note that the retarded solution is singular on the world line of the particle. Following the method previously utilized, we wish to remove this singularity[21], before applying the particle's field to the particle itself. Again, we could note that the retarded Green's function's singularity is contained entirely within

[18] This temporary change should help the reader should they wish to consult the references on this short aside for more information. For the main body of our work the signature is $(+---)$ due to its ease of use in particle physics.

[19] This paper actually contains a mistake, corrected by Hobbs in 1968 [**31**], leading to the absence of a term containing the Ricci tensor in the final equations of motion.

[20] That is, the space-time admits a Cauchy surface - a space-like 3-surface through which every inextendible causal curve in the space-time manifold passes exactly once.

[21] Consequently renormalising the mass.

the symmetric $(G_- + G_-)/2$. Proceeding to subtract this contribution as before, we obtain the equation of motion for a point particle in curved space undergoing electromagnetic radiation reaction. This method was followed by DeWitt and Brehme[22] to obtain

$$m\frac{Du^\mu}{d\tau} = F^\mu_{\text{ext}} + \alpha_c\left(\delta^\mu_\nu + u^\mu u_\nu\right)\left(\frac{2}{3m}\dot{a}^\nu + \frac{1}{3}R^\nu{}_\lambda u^\lambda\right) + 2e^2 u_\nu \int_{-\infty}^{\tau_-} \nabla^{[\mu} G^{\nu]}_{-\lambda'}(X(\tau), X(\tau'))u^{\lambda'} d\tau' \,. \tag{45}$$

The last term is often referred to as the tail term and contains the mentioned dependence on the past history of the world line of the particle. The integral is cut-off at $\tau' = \tau_- - 0^+$ to avoid the singular behaviour of the retarded potential. In flat space, this equation collapses to the Lorentz-Dirac equation. Recalling that this equation is also based on the point particle description, we note the continued presence of the third derivative term and the need for a reduction of order process, or something else, in order to make the description physical.

So far, the extra features of the Green's functions appear only to have manifested themselves in the presence of the tail term. A difficulty is faced, however, in the interpretation of the decomposition of the retarded solution. The combination

$$G_{\text{sym}} = \frac{1}{2}\left(G_- + G_+\right) \,, \tag{46}$$

has the necessary properties that it is symmetric and solves the inhomogeneous wave equation, thus the field that it generates contains the singularity of the retarded field A_-. If we again postulate that the remainder of the field A_- is responsible for the self-force, then although it is indeed regular, this combination has support within both future and past light cones. The appeal of this approach is that there is no support at space-like separation for the arguments

[22]That is, they use the same singular Green's function. The details of their working are based on a definition of the 'direct' and 'tail' contributions to the Green's function as those with support on and within the light cone respectively.

which is in keeping with a field theory perspective. However, taking the effect on the world line itself, then

$$G_- - G_{\mathrm{sym}} = \frac{1}{2}\left(G_- - G_+\right) \,, \tag{47}$$

is dependent on the entire past and future history of the world line, which is somewhat problematic from a causal perspective if we are to then interpret the resultant field as on radiative field acting on the particle. The key to solving this problem, identified by Detweiler and Whiting in 2003 [**17**], is the recognition that although the symmetric combination G_{sym} does indeed contain the singularity, it is not unique in this respect. This non-uniqueness is part of the reason that we stated in the flat space description that the use of $G_- - G_{\mathrm{sym}}$ as wholly responsible for the self-force was *postulated*. We are free to add any solution of the homogeneous wave equation to G_{sym} and the result will remain a solution of the inhomogeneous equation. This then is how we proceed. The additional homogeneous solution is defined precisely to solve the causal issues present in the choice of G_{sym}. We note additionally, that we must ensure that this solution is also symmetric, otherwise we shall affect the symmetric property of the resultant singular solution. We therefore define $H(x, x')$ such that it is equal to the advanced Green's function G_+ when x is in the chronological past of x' and, by symmetry, agrees with G_- when x is in the chronological future of x'. Subtracting this solution from G_{sym}, we define the curved space singular Green's function as

$$G_S = \frac{1}{2}\left(G_- + G_+ - H\right) \,. \tag{48}$$

This function has support at spatially separated points, and the resulting

$$G_R = G_- - G_S \,, \tag{49}$$

is dependent on the history of the world line up to and including the *advanced* time τ_+. Whilst it is somewhat counterintuitive to use a result with apparent dependence at spatial separation, we recall that the decomposition is used only in calculating the effect of the field on the particle itself, hence on the world

line (where the separation is zero). It should be stressed that this choice of Green's function decomposition does not actually affect the resulting equation for the radiation reaction force. It does however, put the interpretation of the Green's function decomposition on a more physically reasonable footing by providing a physical field which can act on the particle.

This treatment of the self-interaction, by identification and subtraction of the singular component of the field, can be extended to other forces as well. Recent work has included the calculation of the self-force for a scalar charge in curved space by Quinn in 2000 [**34**]. In this case, instead of interacting with a vector field, as is the case for electromagnetism, the particle is coupled to a spin-zero scalar field. The equations of motion for a particle with charge q are

$$ma^\mu = q\left(g^{\mu\nu} + u^\mu u^\nu\right)\nabla_\nu \Phi \,, \tag{50}$$

where the scalar field Φ emitted by the particle satisfies the wave equation

$$\left(\Box - \xi R\right)\Phi = -\mu(x)\,, \tag{51}$$

where ξ is a constant measuring coupling to the curvature[23], and where $\mu(x)$ is the charge density given by

$$\mu(x) = q\int_\gamma d\tau\, \delta_4(x, X)\,, \tag{52}$$

on the world line $\gamma = X(\tau)$ and $\delta_4(x, X)$ is the invariant generalisation of the Dirac delta function

$$\delta_4(x, x') = \frac{\delta(x - x')}{\sqrt{-g}}\,. \tag{53}$$

The combination of these equations adds an extra feature to the dynamics in curved space: If one derives the above equations of motion from a variational principle, then the inertial mass must be time-dependent. Specifically,

$$\frac{dm}{d\tau} = -qu^\mu \nabla_\mu \Phi\,. \tag{54}$$

[23]The constant ξ here is arbitrary, however the most commonly picked values are the minimally coupled scalar field with $\xi = 0$ and the conformally invariant $\xi = 1/6$.

Subtracting the singular Green's function from the particle's field, and adding the self interaction to the equations of motion, Quinn found[24]

$$\begin{aligned} ma^\mu &= F^\mu_{\text{ext}} \\ &\quad + \frac{q^2}{4\pi}\left(\delta^\mu_\nu + u^\mu u_\nu\right)\left[\frac{1}{3m}\dot{a}^\nu + \frac{1}{6}R^\nu{}_\lambda u^\lambda + \int_{-\infty}^{\tau_-} d\tau' \nabla^\nu G_-(X(\tau), X(\tau'))\right] \end{aligned} \tag{55}$$

$$\frac{dm}{d\tau} = -\frac{1}{12}\frac{q^2}{4\pi}\left(1 - 6\xi\right)R - \frac{q^2}{4\pi}u^\mu \int_{-\infty}^{\tau_-} d\tau' \nabla_\mu G_-(X(\tau), X(\tau'))\,. \tag{56}$$

The reader can note the similarities, and differences, between the scalar and vector self-force expressions.

Having looked at scalar and vector (electromagnetic) fields, the next type of potential field of interest is that of gravitational radiation. The emission of gravitational radiation is one of the predictions of general relativity and one which is of current interest at the time of writing. A number of detectors have recently been built hoping to receive signals from the gravitational waves reaching earth. From our current perspective, we see the possibility of a point mass interacting with its own gravitational field. We approach this problem analogously to the scalar and vector cases, by considering a point mass and its potential field. As we are considering the gravitational field, the field of the mass will be the perturbation of the space-time that it induces. The idea of a point mass poses some difficulties within general relativity, not to mention the usual difficulties in the consideration of the non-linear equations of motion. However, provided we keep the perturbations produced by the mass small, which we would in any case wish to do given the previous discussions, then we can proceed. We thus consider the case of a small mass moving in a background space-time $g_{\mu\nu}$, which here is analogous to the charged particle moving in the external potential. The unperturbed path is then a geodesic of g. We assume that $g_{\mu\nu}$ is a solution to the vacuum Einstein equations. We then treat the

[24] Quinn's results [**34**] were for the minimally coupled scalar field and were extended by Poisson to arbitrary ξ for his review [**19**].

mass as a perturbation h to this background metric and use the mass itself as the expansion parameter to produce the full perturbed space-time $f_{\mu\nu}$

$$f_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(m^2)\,. \tag{57}$$

As the coupling to gravity, the mass is effectively the charge in this context. In the background space-time, the equations of motion are

$$a^\mu = -\frac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(2h_{\nu\lambda;\kappa} - h_{\lambda\kappa;\nu}\right)u^\lambda u^\kappa\,, \tag{58}$$

using the ';' notation for covariant differentiation. Now, the potential field which we use is not actually that of the perturbation, but rather the trace-reversed tensor γ defined, akin to the Einstein tensor from the Ricci, as

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\left(g^{\lambda\kappa}h_{\lambda\kappa}\right)g_{\mu\nu}\,, \tag{59}$$

where the reader will note the use of the background metric for contraction in keeping with the perturbation approximation. These trace-reversed potentials then satisfy the wave equation

$$\Box\gamma^{\mu\nu} + 2R_{\lambda\ \kappa}^{\ \mu\ \nu}\gamma^{\lambda\kappa} = -16\pi T^{\mu\nu}\,, \tag{60}$$

where $T^{\mu\nu}$ is the stress-energy tensor of the point mass. Following calculations with these potentials, the original h fields can be obtained by trace-reversing again. These equations are the appropriate counterparts for the spin two graviton to the scalar (spin zero) and vector (spin one) potential cases. Subtraction of the singular field from the perturbation leaves the regular, or radiative, field h^R given by

$$h^R_{\mu\nu;\lambda} = -4m\left(u_{(\mu}R_{\nu)\kappa\lambda\rho} + R_{\mu\kappa\nu\rho}u_\lambda\right)u^\kappa u^\rho + h^{\rm tail}_{\mu\nu\lambda}\,, \tag{61}$$

with the tail term given by

$$h^{\rm tail}_{\mu\nu\lambda} = 4m\int_{-\infty}^{\tau_-} d\tau'\,\nabla_\lambda\left(G_{-\mu\nu\mu'\nu'} - \frac{1}{2}g_{\mu\nu}G^{\lambda}_{-\ \ \lambda\mu'\nu'}\right)u^{\mu'}u^{\nu'}\,. \tag{62}$$

The retarded Green's function here is that for the wave equation for γ, hence the trace-reversed presence here. The equations of motion are then

$$a^{\mu} = -\frac{1}{2}\left(g^{\mu\nu} + u^{\mu}u^{\nu}\right)\left(2h^{\text{tail}}_{\nu\lambda\kappa} - h^{\text{tail}}_{\lambda\kappa\nu}\right)u^{\lambda}u^{\kappa}\,, \tag{63}$$

with only the tail term remaining. These equations were found in 1997 by Mino, Sasaki and Tanaka [**32**] and using different methods reproduced by Quinn and Wald [**33**]. Consequently they go by the name 'MiSaTaQuWa equations'. The variety of methods used removes some of the difficulties in the analysis of a point mass. Given that the unperturbed path was the geodesic of the background space-time, from a general relativity perspective, one would naturally ask about the geometric properties of the new path. From the analysis and interpretation of the regular Green's function by Detweiler and Whiting [**17**], we have already noted that the scalar and electromagnetic charges would move under the influence of the combination of the original external field and the particle's radiative field. Thus extending to this case, [**17**] gives us the interpretation of the new path as the geodesic of the space-time with metric

$$f^{R}_{\mu\nu} = g_{\mu\nu} + h^{R}_{\mu\nu}\,, \tag{64}$$

which remains a solution to the vacuum Einstein equations. Work on gravitational radiation reaction uses the above equations, with attempts to calculate the tail terms, for such situations as small black holes and orbits in the Schwarzchild (black hole) metric. One of the aims of such work is to produce a description of the motion in such extreme circumstances and consequently accurately predict the gravitational radiation that the new detectors hope to detect. Should the process work, then we would ultimately obtain a new type of telescope for probing some of the more extreme gravitational events in the cosmos.

The equations of motion for gravitational radiation reaction appear initially somewhat different in form from those for the other fields. Some of this difference is because the above ignores some extra difficulties. Namely, the

above equations are produced using the Lorentz gauge condition $\gamma^{\mu\nu}_{\ ;\nu} = 0$ and are *not* gauge invariant. Under coordinate transformation of the background coordinates using a smooth field of order m, $x^\mu \to x^\mu + \xi^\mu$, the change in the particles acceleration is given by the 'gauge acceleration'

$$\delta a[\xi]^\mu = (\delta^\mu_\nu + u^\mu u_\nu)\left(\frac{D^2\xi^\nu}{d\tau^2} + R^\nu{}_{\lambda\kappa\rho}u^\lambda \xi^\kappa u^\rho\right). \tag{65}$$

The consequences of this, such as the possible gauging away of the self-force accelerations, should indicate the need to add into consideration the full metric perturbation in order to obtain gauge-invariant observables. As we are only giving a brief overview of extensions to the radiation reaction problem here, we shall not go into anymore detail but refer the interested reader to the literature quoted. From this aside, we now return to considerations of flat-space electromagnetic radiation reaction.

4. Quantum Theory

Classical electrodynamics is no longer considered to be the most fundamental theory, but is currently superseded by quantum electrodynamics, or to use the more common acronym, QED. The classical theory is however very successful within limits and forms the basis on which we normally construct the quantum theory, as with many other classical theories. Classical Electromagnetism, unifying two of the fundamental forces of nature, was one of the great success stories of 19th century science. It was in the study of radiation that the cracks began to appear. The ultraviolet catastrophe[25] is usually given as the example of this, whereby classical electromagnetism predicted that a black body at thermal equilibrium would emit radiation with infinite power. This is demonstrably false, with the problem occurring in the short wavelength (hence ultraviolet) region. The well-known solution was Max Planck's quantum hypothesis - that the radiation was emitted only in discrete 'quanta' of energy, which Einstein suggested be used to address the issue. Einstein also

[25]The ultraviolet catastrophe is also known as the Rayleigh-Jeans catastrophe.

used the hypothesis to solve another classical problem relating to radiation: the photoelectric effect. The issue of electromagnetic radiation was thus one at the focus of the early work on quantum theory. Another example frequently presented as a way of showing the successes of early quantum theory is that concerning the structure of the atom and one which is related to our main consideration. After Rutherford's experiments providing the evidence for the positive nucleus model of the atom[26], the orbit style view of the atom, in which the electrons circled the nucleus, like the Newtonian motion of the planets to the sun[27], was the classical model of the motion of the electrons. This model appears to be a fairly good analogy until one considers the oft-ignored radiation reaction and considers the motion of the particle itself. As the electron is continuously accelerated, although with the acceleration vector changing to always point to the nucleus, the theory predicts that it will emit radiation. This would mean that the system would lose energy and thus the prediction is that the electron would spiral into the nucleus, consequently rendering all classical atoms inherently unstable (see Fig. 1.5). This is of course another effect which is (thankfully) demonstrably incorrect. The extension of the quantum hypothesis to the energy spectrum of the atom, thus allowing only certain stable energy levels, quickly gives very accurate predictions. After these beginnings, the full theory of quantum electrodynamics was gradually developed. This has in turn become one of the success stories of 20th century science, and one frequently stated to be the most accurate theoretical model of all time - so far at least.

QED is usually studied using the techniques of perturbation theory, in which the interactions between the particle and electromagnetic fields are expanded as a series in terms of the coupling between them. In Feynman diagrams, the first three terms of the perturbative expansion of the scattering amplitude are given in Fig. 1.6 and are the basic diagrams usually considered

[26]Instead of, say, the plum-pudding model.

[27]Using circled in a more liberal sense to include elliptical motion.

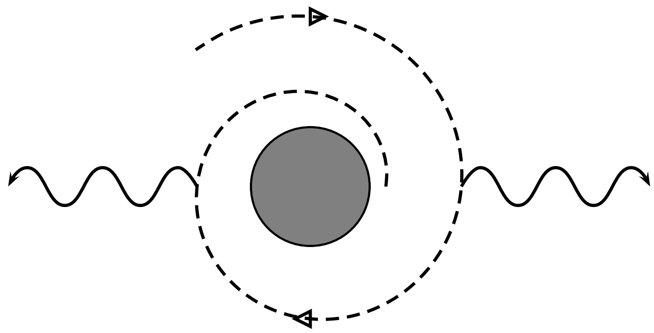

Figure 1.5. An electron in a classical atom would radiate, losing energy, and spiral into the nucleus. It is thus unstable.

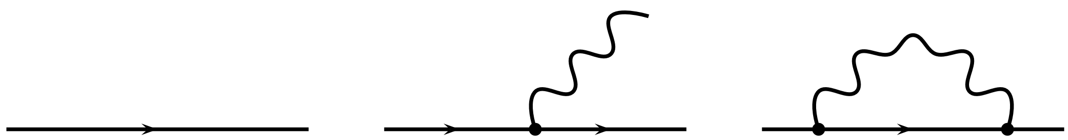

Figure 1.6. The first three types of Feynman diagrams for QED representing the perturbation expansion up to order e^2.

when learning about interactions in quantum field theory.[28] These diagrams are for the perturbation theory up to order e^2, which the reader may recall is the order of the classical radiation reaction. We have a process corresponding to the emission of radiation (the first order interaction). The strength of this

[28]Along with the particle creation/annihilation diagrams, which are really the same process as the emission diagram in Fig. 1.6.

contribution to the perturbation calculation will be dependent on the physical situation i.e. the classical external forces accelerating the particle.[29] The one-loop contribution represents the most basic self-energy process in quantum field theory. It is the self-interaction with a virtual photon emitted and absorbed by the particle itself which we shall also at times term the forward scattering.[30] This represents the contribution of the particle interacting with its own electromagnetic field. The contribution is infinite, as are a number of other self-interaction type processes in QED, hence the reason for the necessity of renormalisation. At this level, renormalisation consists of removing the infinite self-energy contribution in order to obtain finite answers. There is more than one equivalent way in which to achieve this and we shall use the counter-term method in this work as it fits the calculation best. The methods are however all equivalent to redefining the mass.[31]

In the above paragraph, whilst we gave a brief overview of the basic perturbation contributions, the aim was in fact to word the descriptions in terms similar to those that we have been using to describe classical radiation reaction. Some of the similarities in the way in which we deal with both theories should hopefully now be apparent. In both cases, we use a perturbation expansion in terms of the coupling for many calculations.[32] In both cases we can split the processes up into emission and self-energy interactions. In addition, in both the classical and quantum theories it is necessary to subtract an infinite contribution corresponding to the self-energy via a renormalisation of the mass. There are of course also differences; otherwise we would not need to

[29]By classical here we mean that such forces are treated non-perturbatively. Should an external force be added in perturbatively, then the contributions would have to be shown in the diagrams by additional boson lines.

[30]We recall that these diagrams are representative of the contributing terms in the perturbation expansion rather, as is sometimes mistakenly thought, the actual physical process.

[31]For higher order perturbation terms, one would also need to renormalise the field etc.

[32]However, the reasons for using the expansion are different.

replace classical electrodynamics with the quantum theory in the first place. These similarities and differences are then a further motivation for both the work contained here and the details of the models chosen. The above sections of the introduction should be kept in mind when we introduce and justify the model and calculations to be performed.

5. Origins of the present work

This work is based on the initial results and models given in [**5**]. In these papers, Higuchi looks at the process of radiation reaction in quantum mechanics and the non-relativistic approximation in quantum field theory for comparison with the results of classical Abraham-Lorentz-Dirac theory. The papers look at the calculation of the change in position due to radiation reaction, which is labelled the position shift.[33] The comparison is then made between the predicted value of this quantity for the classical theory with the classical limit of the non-relativistic approximation of the first order interactions of quantum field theory for a charged scalar field. The results are that the predictions agree, thus supporting the idea of the Lorentz-Dirac theory as the appropriate classical limit for quantum theory. It is on this base that we build the work presented here. Our aim is to compare the classical and quantum theories of radiation reaction in order to gain a further understanding of the effect in both. Given the debates over the interpretation of the Lorentz-Dirac theory with the associated problems and possible solutions as detailed above, a comparison at the level of the classical limit of the more fundamental QED is also useful. The next sections detail the models used and calculations to be presented along with justifications for the choices made. The models are based on those in [**5**], but extended to a fully relativistic theory, to the spinor field, and also to considerations of the second order interaction at order e^2. The previous sections have detailed the background theories with which

[33]As the model used later is based on that from [**5**], we shall not go into detail now, but rather ask the reader to wait until the next section where the extended model for this work is presented.

we are concerned; the following sections detail and introduce fully the current research on which this work reports.

6. The Model

We wish to compare the effects of radiation reaction in the classical and quantum electrodynamics theories. Possibly the most fundamental effect of radiation reaction is to change the equations of motion. These equations are in turn simply differential equations to be solved for the position of the particle. Consequently, the observable effect due to the existence of radiation reaction, is a (possible) change in the measured position of the particle. We therefore choose to make this observable the measured effect which we shall investigate. To be more precise, we wish to measure the change in position of the particle due to radiation reaction. This rather unwieldy description we give the name the *position shift*.

We now require a model involving radiation reaction in which to make our measurement of the position shift. The reader will recall that the canonical set-up used in the perturbation theory of quantum field theory is the situation in which the fields are regarded to be free at future and past temporal infinity, with the perturbative interaction in between. The particle interpretation is in fact dependent on the states being non-interacting at temporal infinity (past and future). We have a situation in which a free particle enters from past infinity, interacts with the other fields (in our case, the electromagnetic field) and then leaves as a free particle to future infinity. At this point we remind the reader that in the classical theory, the reduction of order procedure, as carried out on the Lorentz-Dirac equation, is equivalent to treating the Lorentz-Dirac force as a perturbation. The two theories we wish to compare are consequently both best represented by the above description of the quantum interaction model. We therefore choose the following: Let the particle travel in a potential which is constant in the asymptotic regions and non-constant for some finite region in between. Only in the non-constant potential region

will the particle experience acceleration and thus radiation reaction. Having given some explanation for the choice of such a model, we now proceed to define precisely the model used for this work.

Let the potential V be dependent on one of the spacetime indices, say x^a. The potential is chosen to be equal to $V_0 = \text{const.}$ for $x^a < X_1^a$ and equal to 0 for $x^a > X_2^a$ for some $X_2^a > X_1^a$. The acceleration is thus non-zero only in the region $X_1^a \leq x^a \leq X_2^a$. The choice of $V = 0$ for the final region is made for simplicity (if it were not, we could simply redefine the potential so that it was). Let us define the three regions as

$$\begin{aligned} \mathcal{M}_- &= \{x | x^a < X_1^a\} \\ \mathcal{M}_I &= \{x | X_1^a \leq x^a \leq X_2^a\} \\ \mathcal{M}_+ &= \{x | x^a > X_2^a\} \,. \end{aligned} \tag{66}$$

With $x^a = x^0$, i.e. a time dependent potential, it is clear that the particle will start in the region with V_0, enter the region of acceleration and thenceforth finish in the region with $V = 0$. For x^a spatial, we require the initial and final momenta to be positive in the x^a direction to achieve the same set-up. The only assumption here is that there is no turning point in that coordinate, something which in fact we shall require later in any case. [34]

We wish to analyse the effect of the radiation reaction which takes place in the region of acceleration as measured by the position shift. It makes sense that the measurement takes place outside the region itself. We note that in the non-interacting region, representing the quantum field by a free field is an approximation which becomes more valid as we move further from the interaction. We thus state that the position shift is measured far enough into the later asymptotic region so that the plane wave approximation for the quantum mode function is accurate. Now, the position shift will be measured by comparing the position of a non-radiation particle, a control particle, to

[34] Strictly speaking we have only so far assumed the weaker condition that the number of turning points is not odd.

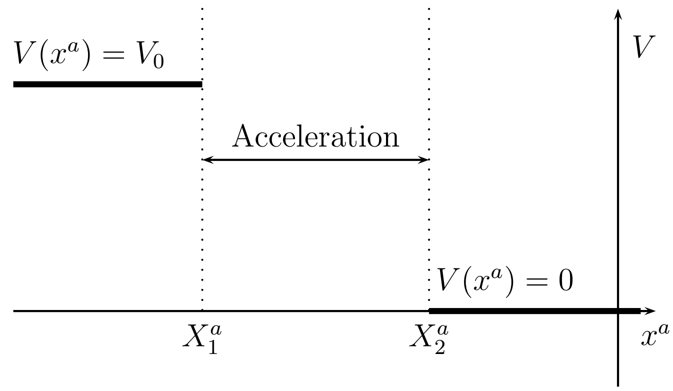

FIGURE 1.7. The potential $V(x^a)$ and period of acceleration.

one undergoing radiation reaction. Again, for the sake of simplicity, we define the coordinate such that the control particle is at the origin at the time of the measurement. This has the added bonus that the position shift is simply the position of the radiating particle at the point of measurement. The positions $X^a_{1,2}$ are thus negative in this coordinate system. Fig. 1.7 represents the model graphically. The choice of $V_0 > 0$ is made here simply for the purpose of the graphical representation.

Now, whilst we wish to treat the radiation reaction effects as a perturbation, we have no particular need to treat the potential as such. In terms of the quantum theory, the potential is treated as a so called classical potential (i.e. non-perturbatively). The potential V is simply the source of the external force which causes the particle to interact with its *own* field. The latter interaction is the one treated perturbatively.

7. Scalar Field

In this section we set up the quantum field theory model of the charged scalar field. We give the appropriate definitions of the field and the conventions and notation which we shall employ in the further discussion. The Lagrangian density $\mathcal{L}$ for the free complex scalar field is given by

$$\mathcal{L} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - (m/\hbar)^2 \varphi^\dagger \varphi \,. \tag{67}$$

From this Lagrangian, the conserved current is given by

$$j_\mu = \frac{i}{\hbar} : \varphi^\dagger \overleftrightarrow{\partial_\mu} \varphi : , \tag{68}$$

where $\overleftrightarrow{\partial_\mu} = \overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu}$. The colons : : represent the normal ordering process which orders creation operators on the left of annihilation operators to ensure that the vacuum expectation value of the current j_μ vanishes. The zeroth component of the conserved current is the charge density given by

$$\rho(x) = \frac{i}{\hbar} : \varphi^\dagger \overleftrightarrow{\partial_t} \varphi : , \tag{69}$$

which we shall have need of in order to calculate the expectation values of the free state.

The equation of motion for a free charged scalar field φ is the Klein-Gordon equation

$$\left(\hbar^2 \Box + m^2\right) \varphi = 0 \, , \tag{70}$$

where $\Box = \partial^\mu \partial_\mu$ is the d'Alembertian operator and m is naturally the mass of the field. In the absence of coupling to another field, φ is expanded via a Fourier decomposition to give

$$\varphi(x) = \hbar \int \frac{d^3\mathbf{p}}{2p_0(2\pi\hbar)^3} \left[A(\mathbf{p}) \Phi_{\mathbf{p}}(x) + B^\dagger(\mathbf{p}) \bar{\Phi}^\dagger_{\mathbf{p}}(x) \right] \, . \tag{71}$$

In this expansion, $\Phi_{\mathbf{p}}(x)$ is the mode function i.e. a solution to the field equation for $\varphi(x)$ (70). Similarly, $\bar{\Phi}_{\mathbf{p}}$ is a solution to the field equation for $\varphi^\dagger(x)$, which for the free field is again (70), thus $\bar{\Phi}_{\mathbf{p}}(x) = \Phi_{\mathbf{p}}(x)$. [35] Modeling the field as a plane wave we substitute

$$\Phi_{\mathbf{p}}(x) = e^{-ip\cdot x/\hbar} \, . \tag{72}$$

$A^\dagger(\mathbf{p})$ and $B^\dagger(\mathbf{p})$ are the creation operators for the positive and negative charged particles, with $A(\mathbf{p})$, $B(\mathbf{p})$ the respective annihilation operators. The

[35] The introduction for the notation $\bar{\Phi}_{\mathbf{p}}$ whilst seemingly superfluous here, shall be needed shortly.

quantisation of the field is given by the commutation relations

$$\left[A(\mathbf{p}), A^\dagger(\mathbf{p}')\right] = \left[B(\mathbf{p}), B^\dagger(\mathbf{p}')\right] = 2p_0(2\pi\hbar)^3\delta^3(\mathbf{p}-\mathbf{p}')\,, \tag{73}$$

with all other commutation relations set to zero. At this point we remind the reader of our conventions as set out above. The measure $d^3\mathbf{p}/p_0$ is a Lorentz invariant element of phase space due to the mass-shell condition $p_0^2 = \mathbf{p}^2 + m^2$. Our convention involves the constant multiplication to give the factors $2p_0(2\pi\hbar)^3$ in both the measure's denominator and in the commutation relations. Note the presence of the $\hbar$'s both here and as an overall multiplier in the field in (71). Their presence is of course frequently omitted in discussions due to the use of natural units ($\hbar = 1$). However, whilst very useful for most particle physics discussions, such a unit system is not conducive to the analysis and investigation of the classical, i.e. $\hbar \to 0$, limit which we shall later wish to perform and hence their inclusion.

The Poincaré invariance of Minkowski space can be employed to define a unambiguous vacuum state $|0\rangle$ for the scalar field, given by the condition $A(\mathbf{p})|0\rangle = B(\mathbf{p})|0\rangle = 0$. The successive applications of the creation operators then build up the Fock space for scalar field with the appropriate particle interpretation.

The free scalar field is used to model the particle in terms of an incoming and outgoing wave packet. We represent the initial state $|i\rangle$ by

$$|i\rangle = \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} f(\mathbf{p}) A^\dagger(\mathbf{p})|0\rangle\,, \tag{74}$$

where the function $f(\mathbf{p})$ is sharply peaked about a given momentum $\overline{\mathbf{p}}$. For our later use, we require that f is sufficiently sharply peaked such that we can approximate $|f(\mathbf{p})|^2$ by $(2\pi\hbar)^3\,[\delta(\mathbf{p}-\overline{\mathbf{p}}) + \mathcal{O}(\hbar^2)]$. The normalisation of the operators $A^\dagger(\mathbf{p})$ is such that the condition $\langle i\,|\,i\rangle = 1$ leads to

$$\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2 = 1\,. \tag{75}$$

This shows that the function $f(\mathbf{p})$ can heuristically be regarded as the one-particle wave function in the momentum representation.

So far, we have dealt only with the free fields i.e. in the absence of the potential. Before considering the interaction between the scalar and electromagnetic fields that will contribute to the radiation reaction process, we must include the external potential V in the presence of which the interaction of interest will take place. As previously stated, we shall treat this external potential non-perturbatively i.e. we shall not expand in orders of V. The inclusion is most easily achieved by substitution of the derivatives as follows:

$$\partial_\mu \varphi \to D_\mu \varphi = \left(\partial_\mu + \frac{i}{\hbar} V_\mu \right) \varphi \,, \tag{76}$$

where the V_μ are the spacetime components of the potential V. In the presence of the potential, the Lagrangian density becomes

$$\mathcal{L} = (D_\mu \varphi)^\dagger D^\mu \varphi - (m/\hbar)^2 \varphi^\dagger \varphi \,. \tag{77}$$

The field equations can be similarly obtained to give

$$\left(\hbar^2 D^\mu D_\mu + m^2 \right) \varphi = 0 \tag{78}$$

$$\left(\hbar^2 D^{\dagger\mu} D_\mu^\dagger + m^2 \right) \varphi^\dagger = 0 \,. \tag{79}$$

In this case we note that the equations are no longer the same; $D_\mu^\dagger \neq D_\mu$. Writing the field in the Fourier mode expansion as before

$$\varphi(x) = \hbar \int \frac{d^3\mathbf{p}}{2p_0(2\pi\hbar)^3} \left[A(\mathbf{p}) \Phi_{\mathbf{p}}(x) + B^\dagger(\mathbf{p}) \bar{\Phi}_{\mathbf{p}}^\dagger(x) \right] \,. \tag{80}$$

The mode functions $\Phi_{\mathbf{p}}(x)$ and $\bar{\Phi}_{\mathbf{p}}(x)$ are solutions to the non-free field equations for $\varphi(x)$ in (78) and $\varphi^\dagger(x)$ in (79) respectively. The difference, arising from the i in D_μ, is of course the charge difference between the particle and antiparticle modes.[36] When analyzing the field in $\mathcal{M}_I$ we shall use the semi-classical expansions for the mode functions, which are detailed in Chapter 2. The commutation relations for the scalar field creation and annihilation operators are those detailed above in (73) for field in the 'free' regions.

[36]The potential has been added using minimal substitution, as the (perturbative) electromagnetic field will be, and thus has a charge coupling in a similar manner.

The next step in the construction of our model is to add the interaction between the scalar and electromagnetic fields, without which there will be no radiation reaction. With the inclusion of coupling to the electromagnetic field, we write the Lagrangian density as

$$\mathcal{L} = \left[(D_\mu + ieA_\mu/\hbar)\,\varphi\right]^\dagger (D^\mu + ieA^\mu/\hbar)\,\varphi - (m/\hbar)^2\varphi^\dagger\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2\,, \tag{81}$$

where A_μ is the electromagnetic potential and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$ is the electromagnetic field tensor. The last term in the Lagrangian density is the Lagrange multiplier representing the choice of the Lorentz gauge condition $\partial_\mu A^\mu = 0$. The choice of the prefactor of $1/2$ on this term is known as the Feynman gauge and is made in order to simplify the photon propagator.[37] We proceed as per the scalar field to give the expansion of the electromagnetic potential in terms of the plane wave solutions viz

$$A_\mu(x) = \int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\left[a_\mu(\mathbf{k})e^{-ik\cdot x} + a_\mu^\dagger(\mathbf{k})e^{ik\cdot x}\right]\,. \tag{82}$$

We make use of the notation $k = |\mathbf{k}|$. Due to the massless nature of the photons, with $k^\mu k_\mu = 0$ and thus $k = k_0$, we will use k and k_0 interchangeably depending on the emphasis required at the time. The quantisation is given by the commutation relations for the photon creation and annihilation operators

$$\left[a_\mu(\mathbf{k}), a_\nu^\dagger(\mathbf{k}')\right] = -g_{\mu\nu}(2\pi)^3 2\hbar k\delta^3(\mathbf{k}-\mathbf{k}')\,. \tag{83}$$

Notice that the scalar field φ is expanded in terms of the momentum $\mathbf{p}$ whereas the electromagnetic field A_μ is expanded in terms of the wave number $\mathbf{k}$. We adopt this convention because the vectors $\mathbf{p}$ and $\mathbf{k}$ are regarded as classical rather than $\mathbf{p}/\hbar$, the wave number of the scalar particle, and $\hbar\mathbf{k}$, the momentum of the electromagnetic field.

We are now in a position to turn our attention to the interaction and evolution of the wave packet taking place during the acceleration period in $\mathcal{M}_I$. The evolution of the state is modeled by perturbation theory. We are

[37] See for example [**24**] for further theoretical details.

interested in terms up to second order in the coupling, i.e. e^2, and consequently need to consider the first two orders in the interaction. The evolution from an initial state $|i\rangle$, written in terms of the interaction Hamiltonian is to second order given by the map

$$|i\rangle \mapsto |i\rangle - \frac{i}{\hbar}\int d^4x\mathcal{H}_I(x)|i\rangle + \left(\frac{-i}{\hbar}\right)^2 \int d^4x\, d^4x' T\left[\mathcal{H}_I(x)\mathcal{H}_I(x')\right]|i\rangle\,, \quad (84)$$

where T is the time ordering operator. The interaction Hamiltonian density is obtained from the interaction Lagrangian to give

$$\mathcal{H}_I(x) = \frac{ie}{\hbar}A_\mu{:}\left[\varphi^\dagger D^\mu\varphi - (D^\mu\varphi)^\dagger\varphi\right]{:}+\frac{e^2}{\hbar^2}\sum_{i=1}^{3} A_iA_i{:}\varphi^\dagger\varphi{:}, \quad (85)$$

where $D_\mu \equiv \partial_\mu + iV_\mu/\hbar$ as before. We have normal-ordered the scalar-field operators to drop the vacuum polarization diagram automatically. Note that the second term is different from what might be naïvely expected, viz $-(e^2/\hbar^2)A_\mu A^\mu : \varphi^\dagger\varphi :$. This difference is due to the presence of interaction terms involving $\dot\varphi$ or $\dot\varphi^\dagger$ in the Lagrangian density.[38]

In addition to the standard one-loop QED process, for scalar QED we also have the contribution where the start and end of the loop are at the same point. In the Feynman diagrams, this is present by the vertex with two photon and two scalar propagators, sometimes known as a seagull vertex, and must be remembered if working from the Feynman rules.[39] The processes contributing to order e^2 from the above interaction Hamiltonian are, in diagrammatic form given in Fig. 1.8. The last two diagrams in Fig. 1.8 jointly give the first non-trivial contribution to the one-particle irreducible Green's function with two external lines, also known as the self energy, which is divergent.

We consequently now come to the renormalisation process to deal with the divergences. To the required order in $\hbar$ for the calculations that shall follow we shall only require the renormalisation of the mass. As we shall be dealing with

[38] The derivation of the interaction Hamiltonian is detailed in appendix B.

[39] For most of the later work, we shall be starting from the operators, and so this contribution will come out of the works on its own. We only use the Feynman rules here for the free-field calculations of the mass counter-terms.

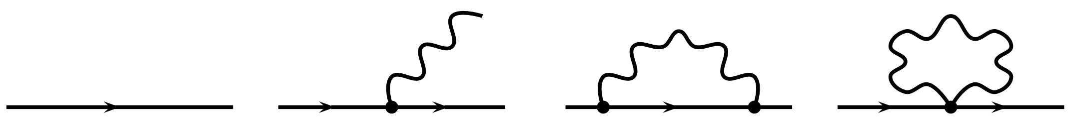

FIGURE 1.8. The Feynman diagrams for scalar QED representing the perturbation expansion up to order e^2.

the contributions from the interaction Hamiltonian terms to the position shift, the natural method of renormalisation will be using the mass counter-term. The contribution of the counter-term, which is of course infinite by definition, can then be added the our results. The counter-term takes the form of the addition to the Lagrangian of

$$\delta\mathcal{L} = \frac{\delta m^2}{\hbar^2}\varphi^\dagger\varphi\,, \tag{86}$$

where the $\hbar^2$ is needed due to our field conventions for the scalar field. The mass counter-term is local, i.e. has no momentum dependence, and is designed to cancel the divergences from the one-loop diagrams for the free-field.[40] The calculation here is thus the standard quantum scalar field theory renormalisation for which the reader is referred to the literature for a full introduction.[41] This standard nature is emphasized due to the fact that in the presence of the potential, the general quantum field theory calculations are *not* standard free field QED, hence the Feynman rules are not used there. The propagator is modified to remove these divergent contributions via the subtraction of the self-energy

$$\frac{i}{p^2-m^2} \mapsto \frac{i}{p^2-m^2-\Sigma(p)}\,, \tag{87}$$

[40]For the scalar field calculations we shall refer jointly to last two diagrams in Fig. 1.8 as the forward scattering or one loop process.

[41]The author recommends, for example [**24**], [**25**] and [**26**].

FIGURE 1.9. The mass counter term contribution to the propagator.

where m here is the bare mass. Rewritten as an expansion

$$\frac{i}{p^2-m^2-\Sigma(p)} = \frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left[-i\Sigma(p)\right]\frac{i}{p^2-m^2} + \frac{i}{p^2-m^2}\left[-i\Sigma(p)\right]\frac{i}{p^2-m^2}\left[-i\Sigma(p)\right]\frac{i}{p^2-m^2} + \dots, \tag{88}$$

it is easy to see that this operation then effectively adds a further Feynman diagram to the perturbation expansion, given in Fig. 1.9. This perturbation contribution can be read straight from the Feynman rules for the scalar field. Using our conventions we obtain, with $K = \hbar k$ for the photon momentum,

$$\begin{aligned} -i\Sigma(p) &= \int \frac{d^4K}{(2\pi)^4}\left\{\left(-\frac{ie}{\hbar}\right)\left[p_\mu + (p_\mu - K_\mu)\right]\right. \\ &\qquad \times \frac{i}{(p-K)^2 - m^2 + i\epsilon}\frac{-i\hbar g^{\mu\nu}}{K^2+i\epsilon}\left(-\frac{ie}{\hbar}\right)\left[p_\nu + (p_\nu - K_\nu)\right] \\ &\qquad \left. + \left(-\frac{ie}{\hbar}\right)^2 \frac{i\hbar\delta^\mu_\mu}{K^2+i\epsilon}\right\} \\ &= -ie^2\int\frac{d^4K}{(2\pi)^4 i}\left\{\frac{(2p-K)^2}{\left[(p-K)^2-m^2+i\epsilon\right]\left[K^2+i\epsilon\right]} - \frac{4}{K^2+i\epsilon}\right\}. \end{aligned} \tag{89}$$

The function Σ is divergent and we need to regularise it, e.g. by dimensional regularisation. Then δm^2 is chosen (as a function of the regularising parameter) to cancel the divergence and ensure that

$$\delta m^2 - \Sigma(p)\big|_{p^2=m_P^2} \to 0 \tag{90}$$

as the regulator is removed, where m_P is the physical mass.

We have now introduced the conventions and definitions for the main components of our quantum theoretic model of the complex scalar field.

8. Spinor Field

In this section we give our definitions and conventions for the quantum model for a spinor field. The spinor field is the spin 1/2 field ψ satisfying the first order Dirac equation

$$\left(i\partial\!\!\!/ - m\right)\psi = 0\,. \tag{91}$$

In the above we have made use of the Feynman 'slash' notation i.e. for the covariant vector A_μ, $A\!\!\!/ := \gamma^\mu A_\mu$. The gamma matrices γ_μ, by virtue of the fact that ψ must also satisfy the Klein-Gordon equation, are subject to the relation $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. In this work we shall make use of the Dirac representation for the γ-matrices (also known as the standard representation and detailed in Appendix C). The Dirac equation is the equation of motion for the field with Lagrangian

$$\mathcal{L} = i\hbar\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi\,, \tag{92}$$

where the barred spinors are defined in terms of γ^0 and the Hermitian conjugate spinor by $\bar{\psi} = \psi^\dagger\gamma^0$. From this Lagrangian, the canonical momentum is given by $\pi(x) = i\psi^\dagger(x)$. Despite the more complicated nature of spinors when compared with a scalar field, the fact that the equations of motion are first order leads to simpler expressions for most basic required quantities. The zeroth component of the current, the charge density, is given by

$$j^0 = \rho(x) =: \psi^\dagger(x)\psi(x) :\,, \tag{93}$$

with the usual normal ordering.

The free spinor field is expanded in the following way

$$\psi(t, \mathbf{x}) = \int \frac{d^3p}{(2\pi\hbar)^3}\frac{m}{p_0}\sum_\alpha \left[b_\alpha(\mathbf{p})\Phi_\alpha(\mathbf{p}) + d^\dagger_\alpha(\mathbf{p})\Psi_\alpha(\mathbf{p})\right]\,. \tag{94}$$

The expansion includes the sum over α, the spin index. The spin 1/2 field will have its spin along a particular axis in one of two states, 'up' and 'down', which shall be represented in the appropriate solution to (91). The different spin

states have their own creation and annihilation operators. These operators satisfy the anticommutation relations

$$\begin{aligned}
\{b_\alpha(\mathbf{p}), b^\dagger_\beta(\mathbf{p}')\} &= \{d_\alpha(\mathbf{p}), d^\dagger_\beta(\mathbf{p}')\} = \frac{p_0}{m}(2\pi\hbar)^3\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\alpha\beta} \\
\{b_\alpha(\mathbf{p}), b_\beta(\mathbf{p}')\} &= \{b^\dagger_\alpha(\mathbf{p}), b^\dagger_\beta(\mathbf{p}')\} = 0 \\
\{d_\alpha(\mathbf{p}), d_\beta(\mathbf{p}')\} &= \{d^\dagger_\alpha(\mathbf{p}), d^\dagger_\beta(\mathbf{p}')\} = 0\,. \qquad (95)
\end{aligned}$$

We recall that quantisation of the spinor field uses the anticommutation relations, as opposed the commutation relations, to ensure that the energy of the field is positive definite. This also means that when normal ordering one must be careful to take the appropriate minus signs when swapping the order of fields. Here we have used the multiple p_0/m in the denominator of the measure and the anticommutation relations.

From the scalar definitions, we recall that there was no need in the free field case to distinguish between solutions of the field equations for the field and its conjugate. However, the distinction was important when adding the potential. The same should be considered here. The conjugate of the Dirac equation gives the field equations for the barred-conjugate field

$$\bar{\psi}(x)\left(i\hbar\overleftarrow{\not\partial}+m\right)=0\,, \qquad (96)$$

where the arrow indicates that ∂ acts on those terms to the left (i.e. on the $\bar{\psi}(x)$ field here). We consequently regard the mode function Φ as a solution for the Dirac equation for ψ and $\bar{\Psi}$ as a solution of the conjugate equation for $\bar{\psi}$. [42] The latter designation is merely for emphasis; the mode function Ψ is still a solution to the Dirac equation. However, we wish to emphasize that the mode functions are not conjugate to each other. This is in keeping with the definitions of the scalar mode functions and again, for the free field the distinction is irrelevant. The functions $\Phi_\alpha(\mathbf{p}), \Psi_\alpha(\mathbf{p})$ for the free field plane

[42] Note that $\bar{\Psi}$ is the barred-conjugate of the mode function in the decomposition (94), which must be a vector in the same vector space as Φ.

wave solution are given by

$$\Phi^\alpha(\mathbf{p}) = u_\alpha(p)e^{-ip\cdot x/\hbar} \tag{97}$$

$$\Psi^\alpha(\mathbf{p}) = v_\alpha(p)e^{ip\cdot x/\hbar}\,. \tag{98}$$

From the Dirac equation, the spinors satisfy the equations

$$(\not{p} - m)u(p) = 0 \tag{99}$$

$$(\not{p} + m)v(p) = 0\,. \tag{100}$$

The spinors $u_\alpha(p), v_\alpha(p)$ are given by

$$u_\alpha(p) = \sqrt{\frac{p_0+m}{2m}}\begin{pmatrix} s_\alpha \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}s_\alpha \end{pmatrix} \tag{101}$$

$$v_\alpha(p) = \sqrt{\frac{p_0+m}{2m}}\begin{pmatrix} \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}s_\alpha \\ s_\alpha \end{pmatrix}\,. \tag{102}$$

For spin up/down along the x^i axis, the vectors s_α are the corresponding two eigenvectors of the spin matrix σ^i. To simplify the notation, we shall make use of an Einstein convention on the spin indices, for which we shall reserve the early-alphabet Greek letters $\alpha, \beta, \gamma, \delta$. Thus $b_\alpha\Phi^\alpha = \sum_{\alpha=1,2} b_\alpha\Phi_\alpha$. The mid-alphabet Greek letters μ, ν etc. will be reserved for the spacetime indices which satisfy the usual Einstein convention with space-time metric convention $(+---)$. Latin letters denote space indices only.

We use the free field to model the wave packet in the non-interacting regions. Similarly to before, we represent the incoming wave packet as a distribution heuristically regarded as the one-particle wave function in the momentum representation:

$$|i\rangle = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\sqrt{\frac{m}{p_0}}f(\mathbf{p})b^\dagger_\alpha(\mathbf{p})|0\rangle\,, \tag{103}$$

where f is sharply peaked about the initial momentum in the region $\mathcal{M}_-$ and normalised via $\langle i|i\rangle = 1$

$$\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}f^*(\mathbf{p})f(\mathbf{p}) = 1\,. \tag{104}$$

Having considered the field in the asymptotic regions, we must now consider the field in the interaction region $\mathcal{M}_I$ in the presence of the potential V. We proceed in the same way as previously by introducing the potential via the transformation of the derivative

$$\partial_\mu \psi \mapsto D_\mu \psi = \left(\partial_\mu + \frac{i}{\hbar} V_\mu \right) \psi \,. \tag{105}$$

We again stress that we treat the potential non-perturbatively. The Lagrangian is now

$$\mathcal{L} = i\hbar\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi \,. \tag{106}$$

The relative minus sign on V in the conjugate of (105) ultimately represents the opposite charge of the antiparticle solutions. Let $\not{D} = \gamma^\mu D_\mu$. The appropriate equations of motion are now

$$\left(i\hbar\not{D} - m \right) \psi = 0 \tag{107}$$

and the conjugate gives

$$\bar{\psi} \left(i\hbar \overleftarrow{\not{D}}^{\dagger} + m \right) = 0 \,, \tag{108}$$

where the arrow indicates the differentiation of term to the left. We note that $\overline{(\not{D}\psi)} = \bar{\psi}\not{D}^{\dagger}$, leading to the second equation. The mode functions in the interacting region are now solutions of these two equations i.e.

$$\left(i\hbar\not{D} - m \right) \Phi_\alpha(x) = 0 \tag{109}$$

$$\bar{\Psi}_\alpha(x) \left(i\hbar \overleftarrow{\not{D}}^{\dagger} + m \right) = 0 \,. \tag{110}$$

We add the electromagnetic field via minimal substitution as before, which in this case gives

$$\not{D} \rightarrow \not{D} + ie\not{A}/\hbar \,, \tag{111}$$

where once again the electromagnetic field has the expansion

$$A_\mu(x) = \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \left[a_\mu(\mathbf{k}) e^{-ik\cdot x} + a^\dagger_\mu(\mathbf{k}) e^{ik\cdot x} \right] \,, \tag{112}$$

with the commutation relations

$$\left[a_\mu(\mathbf{k}), a_\nu^\dagger(\mathbf{k}')\right] = -g_{\mu\nu}(2\pi)^3 2\hbar k \delta^3(\mathbf{k}-\mathbf{k}') \,. \tag{113}$$

Most of what was said previously about the details of the EM field applies equally here. The QED Lagrangian, in the presence of the classical potential V, is given by

$$\mathcal{L} = i\hbar\bar{\psi}\gamma^\mu \left(D_\mu + \frac{ie}{\hbar}A_\mu\right)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 \,. \tag{114}$$

The interaction Lagrangian can be given from (114) by

$$\mathcal{L}_I = -e\bar{\psi}\not{A}\psi \,. \tag{115}$$

Unlike the scalar case, the switch to the Hamiltonian formulation is straightforward and we find that the interaction Hamiltonian is simply to negative of $\mathcal{L}_I$ viz

$$\mathcal{H}_I = e : \bar{\psi}\not{A}\psi : , \tag{116}$$

where we have added the normal ordering. The interaction Hamiltonian is then substituted as appropriate in the evolution of the state. The evolution in (84) is a general statement of perturbation theory and thus relevant here:

$$|i\rangle \to |i\rangle - \frac{i}{\hbar}\int d^4x \mathcal{H}_I(x)|i\rangle + \left(\frac{-i}{\hbar}\right)^2 \int d^4x \, d^4x' T\left[\mathcal{H}_I(x)\mathcal{H}_I(x')\right]|i\rangle \,. \tag{84}$$

We again look at the perturbation expansion contributions to order e^2. As with the scalar case, we have null, emission and forward scattering processes. For the spinor fields, which is of course standard QED, the forward scattering does not contain the second circular loop process seen as the last process in Fig. 1.8, as there is no seagull vertex. We instead simply have the three diagrams described in the introduction and given in Fig. 1.6, which we repeat in this section (Fig. 1.10) to aid the reader. The remaining one loop diagram is still divergent and the contribution is subtracted via renormalisation in much the

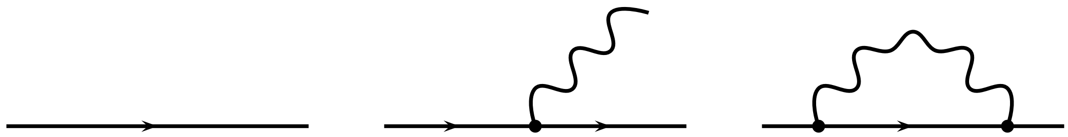

FIGURE 1.10. The first three types of Feynman diagrams for QED representing the perturbation expansion up to order e^2.

same way as briefly described in the scalar field section. To order e^2 the mass counter term adds to the Lagrangian the additional term

$$\delta\mathcal{L} = \delta m\bar{\psi}\psi\,. \tag{117}$$

The counterterm δm is again local, i.e. has no momentum dependence. The spinor propagator, written in terms of the bare mass m, is modified to remove the divergences from the one-loop contribution via

$$\begin{aligned}\frac{i}{\not{p}-m} \to{}& \frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}\left[-i\Sigma(p)\right]\frac{i}{\not{p}-m} \\ &+ \frac{i}{\not{p}-m}\left[-i\Sigma(p)\right]\frac{i}{\not{p}-m}\left[-i\Sigma(p)\right]\frac{i}{\not{p}-m} + \dots \\ ={}& \frac{i}{\not{p}-m-\Sigma(p)}\,.\end{aligned} \tag{118}$$

The self-energy $\Sigma(p)$ is given here analogously to the situation described for the scalar field and can similarly be represented by an additional Feynman diagram contribution (see Fig. 1.9).[43] Using the Feynman rules for standard (spinor) QED applied to the one loop diagram, we obtain

$$\Sigma(p) = -\frac{ie^2}{\hbar}\int\frac{d^4K}{(2\pi)^4}\frac{g_{\mu\nu}}{K^2+i\epsilon}\gamma^\mu\frac{1}{\not{p}-\not{K}-m+i\epsilon}\gamma^\nu\,. \tag{119}$$

The self-energy is again divergent and may proceed as in the scalar case to regularise it via dimensional regularisation. Similarly to the previous case, the

[43] We are approaching these fields in a somewhat reverse order by giving the '*standard*' QED results second, due to the order they are used in this work.

counterterm δm is then chosen, as a function of the regularising parameter, to cancel the divergence so that as the regulator is removed we have

$$\delta m - \Sigma(p)|_{\not{p}=m_P} \to 0\,, \tag{120}$$

where m_P is the physical mass.

This concludes our introduction to the model that we shall use for the quantum field theory description of the Dirac spinor field for QED. An exhaustive or pedagogic introduction to quantum field theory would be out of place here and the reader unfamiliar with the canonical descriptions outlined above is referred the one of the many textbooks, or indeed courses, designed specifically for that purpose eg. [**25**], [**26**] or [**24**]. On the other hand, the above two sections should now provide a reader familiar with quantum field theory with an appropriate reference for the definitions and conventions that are used in the rest of this work.

CHAPTER 2

Semiclassical Approximation

In this chapter we introduce and calculate the semiclassical approximations to be used to model the scalar and spinor quantum fields during their interaction with the classical potential.

1. Semiclassical and WKB approximations

In this section we have two purposes to keep in mind. Firstly, we need to solve the field equations to find expressions for the mode functions during the period of acceleration. Secondly, we aim to take the classical limit i.e. the limit in which $\hbar \to 0$. It is thus appropriate to use a semiclassical expansion, i.e. an expansion in terms of $\hbar$, in order to obtain our mode function solutions. Due to the nature of the model, we shall not however be solving the equations exactly in terms of known quantities. This is a simple consequence of the fact that we do not wish to constrain the possible behaviour any more than is absolutely necessary. So far little has been said of any possible constraints on the acceleration. Most of the constraints that will become apparent are in fact due to the semiclassical expansion detailed in this section. In order for the expansion to be valid, and indeed found by the following method, some restraints are necessary.

In order to set the scene before presenting the relevant calculations, let us briefly recall some of the basic theory of semiclassical expansions in quantum theory. The expansion of the wave function in orders of $\hbar$ in quantum

mechanics goes by the name of the WKB approximation, named after Wentzel-Kramers-Brillouin from their 1926 development of the method.[1] Strictly speaking, the WKB approximation is the expansion up to order $\hbar$ of the solution to the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x})\right]\psi(\mathbf{x},t)\,. \tag{121}$$

The complex solution $\psi(\mathbf{x},t)$ to (121), rewritten in terms of some function S as $e^{iS(\mathbf{x},t)/\hbar}$ leads, with the assumption $\psi \neq 0$, to

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\nabla S\right)^2 - \frac{i\hbar}{2m}\nabla^2 S + V\,. \tag{122}$$

The formal 'classical limit', $\hbar \to 0$, gives the Hamilton-Jacobi equation

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\nabla S\right)^2 + V\,. \tag{123}$$

In general, the semiclassical expansion is the expansion of S in terms of $\hbar$ viz

$$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \ldots\,. \tag{124}$$

With the substitution of this expansion,the appropriate equation of motion can then be solved order by order. For the Schrödinger equation we have, for S_0 and S_1,

$$-\frac{\partial S_0}{\partial t} = \frac{1}{2m}\left(\nabla S_0\right)^2 + V \tag{125}$$

$$-\frac{\partial S_1}{\partial t} = \frac{1}{2m}\left[-i\nabla^2 S_0 + 2\boldsymbol{\nabla} S_0\cdot\boldsymbol{\nabla} S_1\right]\,. \tag{126}$$

Note that the equation for S_0 is again the Hamilton-Jacobi equation. As an example, consider the time-independent *one dimensional* Schrödinger equation. For a wavefunction proportional to $e^{-iEt/\hbar}$, we note that $S(x,t) = S(x) - Et$, and so we may separate out the $e^{-iEt/\hbar}$ factor and consider the semiclassical

[1]This is one of those cases where multiple names are sometimes used in attempts to credit the correct people. The WKB approximation is also known as the WKBJ. The J is for Harold Jeffreys, who in 1923 developed the general method of approximating linear, second-order differential equations, including the later (1925) Schrödinger equation. Early quantum mechanics texts also use WBK, BWK, WKBJ and BWKJ.

expansion as terms dependent on $\mathbf{x}$ only: $S(\mathbf{x}) = S_0(\mathbf{x}) + \hbar S_1(\mathbf{x}) + \ldots$. The solution up to order $\hbar$ in this expansion is

$$\psi(x,t) = \frac{C}{\sqrt{p(x)}} \exp\left[\pm\frac{i}{\hbar}\int p(x')dx'\right] e^{-iEt/\hbar}, \tag{127}$$

where $p(x) = \sqrt{2m(E - V(x))}$ is the classical momentum of the particle and C is a constant.[2] This example shows explicitly the general restriction on the validity of this approximation, namely that it breaks down when the classical particle reaches a turning point, i.e. $p(x) = 0$ above.

This conclusion can also be reached by analysis of the validity of the approximation itself. In order for us to be justified in taking the $\hbar$ expansion then the truncated series that we use must be a good approximation. From the notation above, we would require that $\hbar S_1$ be much smaller than S_0. From the equation (122), we require that the $\hbar$ term be much smaller than the other $\hbar^0$ terms. These general requirements give in our current context the condition

$$|\,(\nabla S)^2\,| \gg \hbar|\nabla^2 S|\,. \tag{128}$$

If we turn to our specific example of the time-independent Schrödinger equation, we obtain

$$p(x)^2 \gg \hbar\left|\frac{dp(x)}{dx}\right|\,. \tag{129}$$

Substituting the definition of $p(x)$ from above and rearranging, we find

$$\left|\frac{\hbar dV(x)/dx}{2(E - V(x))p(x)}\right| \ll 1\,. \tag{130}$$

As stated, we thus arrive at the same conclusion regarding the validity conditions i.e. that the approximation breaks down at the classical turning point $E = V$.[3] One should however note that the approximation may still be valid *beyond* the classical turning point. This encapsulates the fact that in quantum mechanics the probability amplitude need not be zero in the classically

[2]The steps of this calculation are nearly identical to those for scalar field which we shall present fully later. As we present the Schrödinger results as a motivational example we have omitted the details here.

[3]Recall that $p(x) = 0$ here too.

forbidden regions, hence providing for quantum phenomena such as quantum tunnelling. The above condition, it should be recalled, is one for the approximation, rather than the quantum wavefunction itself. That the WKB approximation does not break down in the classically forbidden, yet quantum-allowed, regions is an important point which demonstrates that the $\hbar \to 0$ limit of the semiclassical expansion may still contain quantum phenomena and thus can not technically be assumed to be the classical limit, in the sense of producing the purely classical theory. This limit is nevertheless frequently referred to as the classical limit, and as we shall use this limit to compare the quantum theory effects with those of the classical theory, it shall be referred to as such here with the above caveat to be kept in mind.

Having now reminded ourselves of the canonical semiclassical theory for quantum mechanics, we can now turn our attention to the approximations needed for the quantum model we have set out. We start by looking at the semiclassical approximation for the scalar field and then consider the same for the Dirac spinor field.

2. Semiclassical Scalar solutions

In this section we consider the scalar field solutions to the Klein-Gordon field equations in the presence of the potential. In the region of the acceleration of the particle, the mode functions are these solutions to the field equations. We desire a semiclassical expansion of the mode functions. Firstly, let us consider the case of the time dependent (and space independent) potential: $V = (0, \mathbf{V}(t))$ with the gauge choice $V_0 = 0$. In this case we will have conservation of momentum. Firstly, we separate he mode function as follows:

$$\Phi_{\mathbf{p}} \propto e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \phi_{\mathbf{p}}(t) \,. \tag{131}$$

The wave equation that is satisfied by $\Phi_{\mathbf{p}}$ is

$$\begin{aligned}\big[\hbar^2\partial_t^2 + (-i\hbar\partial_x - V_x(t))^2 + (-i\hbar\partial_y - V_y(t))^2 \\ + (-i\hbar\partial_z - V_z(t))^2 + m^2\big]\,\Phi_p = 0\,,\end{aligned} \tag{132}$$

thus the equation that must be satisfied by $\phi_{\mathbf{p}}(t)$ is

$$\big[\hbar^2\partial_t^2 + (p_x - V_x(t))^2 + (p_y - V_y(t))^2 + (p_z - V_z(t))^2 + m^2\big]\,\phi_{\mathbf{p}}(t) = 0\,. \tag{133}$$

We now wish to find the semiclassical expansion of this solution. In the scalar case, we shall need to take only the first two terms of an expansion in $\hbar$ in the exponential, which translates to order $\hbar^0$ for the mode function, viz

$$\phi_{\mathbf{p}}(t) = \exp\left[-\frac{i}{\hbar}S^{(0)} - S^{(1)}\right]\,. \tag{134}$$

We substitute this expression into the wave equation and then equate order by order. The first term, the t differential gives

$$\begin{aligned}&\hbar^2\partial_t^2 \exp\left(-\frac{i}{\hbar}S^{(0)} - S^{(1)}\right) \\ &= \hbar^2\partial_t\left[\exp\left(-\frac{i}{\hbar}S^{(0)} - S^{(1)}\right)\left[-\frac{i}{\hbar}\partial_t S^{(0)} - \partial_t S^{(1)}\right]\right] \\ &= \left\{-\left(\partial_t S^{(0)}\right)^2 + 2i\hbar\partial_t S^{(0)}\partial_t S^{(1)} - i\hbar\partial_t^2 S^{(0)} - \hbar^2\partial_t^2 S^{(1)} + \hbar^2\left(\partial_t S^{(1)}\right)^2\right\}\phi_{\mathbf{p}}(t)\,.\end{aligned} \tag{135}$$

Thus the order $\hbar^0$ terms give an equation for $S^{(0)}$

$$\left(\partial_t S^{(0)}\right)^2 = (p_x - V_x(t))^2 + (p_y - V_y(t))^2 + (p_z - V_z(t))^2 + m^2\,, \tag{136}$$

which we can solve to give

$$S^{(0)} = \int_0^t E_p(t')dt'\,, \tag{137}$$

where

$$E_p(t) = \sqrt{(p_x - V_x(t))^2 + (p_y - V_y(t))^2 + (p_z - V_z(t))^2 + m^2}\,. \tag{138}$$

This is the classical energy of the particle.[4] The order $\hbar^1$ terms give

$$2\partial_t S^{(0)}\partial_t S^{(1)} = \partial_t^2 S^{(0)}\,, \tag{139}$$

which has the solution

$$\begin{aligned} S^{(1)} &= \int_0^t \frac{\partial_{t'}^2 S^{(0)}}{2\partial_{t'} S^{(0)}} dt' \\ &= \frac{1}{2}\int d\left(\partial_t S^{(0)}\right)\frac{1}{\partial_t S^{(0)}} \\ &= \frac{1}{2}\ln E_p(t) + \text{const.} \end{aligned} \tag{140}$$

Thus the t-dependent part of the wave function is given by

$$\phi_{\mathbf{p}}(t) = \frac{C}{\sqrt{E_p}} e^{-i\int E_p dt/\hbar} \tag{141}$$

$$= \sqrt{\frac{p_0}{E_p(t)}}\exp\left[-\frac{i}{\hbar}\int_0^t E_p(\zeta)d\zeta dt'\right]\,, \tag{142}$$

where $p_0 = \sqrt{\mathbf{p}^2 + m^2}$. The case of the potential dependent on one of the spatial coordinates can be given by considering one example. Here we choose a z dependent potential $V(z) = (V_t(z), V_x(z), V_y(z), 0)$ with the gauge choice $V_z = 0$. Again, we separate out the constituent parts of the mode function, this time producing

$$\Phi_{\mathbf{p}} = \phi_{\mathbf{p}}(z) e^{-\frac{i}{\hbar}(p_0 t - p_x x - p_y y)}\,. \tag{143}$$

The wave equation that is satisfied by $\Phi_{\mathbf{p}}$ is

$$\begin{aligned} &\left[-\left(-i\hbar\partial_t - V_t(z)\right)^2 + \left(-i\hbar\partial_x - V_x(z)\right)^2\right. \\ &\qquad \left. + \left(-i\hbar\partial_y - V_y(z)\right)^2 + \left(-i\hbar\partial_z\right)^2 + m^2\right]\Phi_p = 0\,, \end{aligned} \tag{144}$$

and consequently that for $\phi_{\mathbf{p}}(z)$ is

$$\left[-(p_0 - V_t(z))^2 + \left(p_x - V_x(z)\right)^2 + \left(p_y - V_y(z)\right)^2 + \left(-i\hbar\partial_z\right)^2 + m^2\right]\phi_{\mathbf{p}}(z) = 0\,. \tag{145}$$

[4] We have labelled the subscript as p to distinguish which momentum this energy is related too. It should be noted that $E_p(t)$ is dependent on the vector $\mathbf{p}$.

As with the previous case, we expand to order $\hbar^0$ overall, viz

$$\phi_{\mathbf{p}}(z) = \exp\left[\frac{i}{\hbar}S^{(0)} + S^{(1)}\right] . \tag{146}$$

Proceeding to analyse the solution order by order we note that the z differential gives

$$\begin{aligned}
&- \hbar^2\partial_z^2 \exp\left(\frac{i}{\hbar}S^{(0)} + S^{(1)}\right) \\
&= -\hbar^2\partial_z\left[\exp\left(\frac{i}{\hbar}S^{(0)} + S^{(1)}\right)\left[\frac{i}{\hbar}\partial_z S^{(0)} + \partial_z S^{(1)}\right]\right] \\
&= -\left\{-\left(\partial_z S^{(0)}\right)^2 + 2i\hbar\partial_z S^{(0)}\partial_z S^{(1)} + i\hbar\partial_z^2 S^{(0)} + \hbar^2\partial_z^2 S^{(1)} + \hbar^2\left(\partial_z S^{(1)}\right)^2\right\} \\
&\qquad \times \phi_{\mathbf{p}}(z) .
\end{aligned} \tag{147}$$

Thus the order $\hbar^0$ terms again produce an equation for $S^{(0)}$

$$\left(\partial_z S^{(0)}\right)^2 = (p_0 - V_t(z))^2 - (p_x - V_x(z))^2 - (p_y - V_y(z))^2 - m^2 , \tag{148}$$

which solves to give

$$S^{(0)} = \int_0^z \kappa_p(z')dz' , \tag{149}$$

where

$$\kappa_p(z) = \sqrt{(p_0 - V_t(z))^2 - (p_x - V_x(z))^2 - (p_y - V_y(z))^2 - m^2} . \tag{150}$$

The order $\hbar^1$ terms give

$$2\partial_z S^{(0)}\partial_z S^{(1)} = -\partial_z^2 S^{(0)} , \tag{151}$$

which has the solution

$$\begin{aligned}
S^{(1)} &= -\int_0^z \frac{\partial_{z'}^2 S^{(0)}}{2\partial_{z'} S^{(0)}} dz' \\
&= -\frac{1}{2}\int d\left(\partial_z S^{(0)}\right)\frac{1}{\partial_z S^{(0)}} \\
&= -\frac{1}{2}\ln\kappa_p(z) + \text{const.}
\end{aligned} \tag{152}$$

Thus the z-dependent part of the wave function is given by

$$\begin{aligned}\phi_{\mathbf{p}}(z) &= \frac{C}{\sqrt{\kappa_p}} e^{i\int \kappa_p dz/\hbar} \\ &= \sqrt{\frac{p_z}{\kappa_p(z)}} \exp\left[\frac{i}{\hbar}\int_0^z \kappa_p(\zeta)d\zeta\right], \end{aligned} \tag{153}$$

with $p_z = \sqrt{p_0^2 - p_x^2 - p_y^2 - m^2}$. The extension to potentials dependent on x or y is straightforward and by simple substitution, hence not repeated here.

2.1. Antiparticle mode functions. We recall that the antiparticle mode functions $\bar{\Phi}_{\mathbf{p}}(x)$ are solutions to the wave equation

$$\left(\hbar^2 D^{\dagger\mu} D^{\dagger}_{\mu} + m^2\right)\varphi^{\dagger} = 0\,. \tag{154}$$

Transformation between the particle/antiparticle solutions is thus accomplished by the transformation $V \to -V$. For the time-dependent potential, we thus have

$$\bar{\Phi}_{\mathbf{p}}(x) = \bar{\phi}_{\mathbf{p}}(t) e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}, \tag{155}$$

with

$$\bar{\phi}_{\mathbf{p}}(t) = \sqrt{\frac{p_0}{E_{p+}(t)}} \exp\left[-\frac{i}{\hbar}\int_0^t E_{p+}(\zeta)d\zeta\right], \tag{156}$$

where

$$E_{p+}(t) = \sqrt{|\mathbf{p} + \mathbf{V}(t)|^2 + m^2}\,. \tag{157}$$

Similarly for the potential dependent on the spatial coordinate z (for example) we have

$$\bar{\Phi}_{\mathbf{p}}(x) = \bar{\phi}_{\mathbf{p}}(z) e^{-\frac{i}{\hbar}(p_0 t - p_x x - p_y y)}, \tag{158}$$

with

$$\bar{\phi}_{\mathbf{p}}(z) = \sqrt{\frac{p_z}{\bar{\kappa}(z)}} \exp\left[\frac{i}{\hbar}\int_0^z \kappa_{p+}(\zeta)d\zeta\right], \tag{159}$$

where

$$\kappa_{p+}(z) = \sqrt{(p_0 + V_t(z))^2 - (p_x + V_x(z))^2 - (p_y + V_y(z))^2 - m^2}\,. \tag{160}$$

3. Semiclassical Spinor solutions

3.1. Positive Energy Solution. In a time-dependent potential the mode function can be split into its space and time dependent parts:

$$\Phi(x) = \psi(t)e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}, \tag{161}$$

where we write the time-dependent component as

$$\psi(t) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \exp\left(-\frac{i}{\hbar}S\right), \tag{162}$$

with the semiclassical expansion contained within the spinor term:

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi^{(0)} \\ \chi^{(0)} \end{pmatrix} + \hbar \begin{pmatrix} \varphi^{(1)} \\ \chi^{(1)} \end{pmatrix} + \hbar^2 \begin{pmatrix} \varphi^{(2)} \\ \chi^{(2)} \end{pmatrix} + \dots . \tag{163}$$

This mode function must obey the Dirac equation with a time-dependent potential. Defining $\tilde{\mathbf{p}} = \mathbf{p} - \mathbf{V}(t)$, we have

$$\begin{aligned} i\hbar\partial_t\Phi(x) - [\boldsymbol{\alpha}\cdot(-i\hbar\nabla - \mathbf{V}(t)) + \beta m]\,\Phi(x) &= 0\,, \\ i\hbar\partial_t\psi(t) - [\boldsymbol{\alpha}\cdot\tilde{\mathbf{p}} + \beta m]\,\psi(t) &= 0\,. \end{aligned} \tag{164}$$

Hence

$$\begin{aligned} i\hbar \begin{pmatrix} \dot\varphi \\ \dot\chi \end{pmatrix} \exp\left(-\frac{i}{\hbar}S\right) + i\hbar \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \left(-\frac{i}{\hbar}\dot S\right) \exp\left(-\frac{i}{\hbar}S\right) & \\ - [\boldsymbol{\alpha}\cdot\tilde{\mathbf{p}} + \beta m] \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \exp\left(-\frac{i}{\hbar}S\right) &= 0\,. \end{aligned} \tag{165}$$

Substituting the $\hbar$ expansion (163) into this equation, we obtain at lowest order

$$\dot S \begin{pmatrix} \varphi^{(0)} \\ \chi^{(0)} \end{pmatrix} = [\boldsymbol{\alpha}\cdot\tilde{\mathbf{p}} + \beta m] \begin{pmatrix} \varphi^{(0)} \\ \chi^{(0)} \end{pmatrix}. \tag{166}$$

Defining $E = \sqrt{\tilde{\mathbf{p}}^2 + m^2}$, the eigenvalues of the matrix $\boldsymbol{\alpha}\cdot\tilde{\mathbf{p}} + \beta m$ are $\pm E$. In this section we are considering the two 'positive energy' mode functions (i.e.

those solutions corresponding to the $+E$ eigenvalue). We obtain

$$S = \int_0^t E(\xi) d\xi \,, \tag{167}$$

and

$$\chi^{(0)} = \frac{\boldsymbol{\sigma} \cdot \tilde{\mathbf{p}}}{E+m} \varphi^{(0)} \,. \tag{168}$$

At higher orders, we have

$$i \begin{pmatrix} \dot{\varphi}^{(n)} \\ \dot{\chi}^{(n)} \end{pmatrix} + \begin{pmatrix} E-m & -\boldsymbol{\sigma} \cdot \tilde{\mathbf{p}} \\ -\boldsymbol{\sigma} \cdot \tilde{\mathbf{p}} & E+m \end{pmatrix} \begin{pmatrix} \varphi^{(n+1)} \\ \chi^{(n+1)} \end{pmatrix} = \mathbf{0}, \quad \forall n = 0, 1, 2, 3, \ldots \,. \tag{169}$$

Multiplying both sides by

$$\begin{pmatrix} E+m & \boldsymbol{\sigma} \cdot \tilde{\mathbf{p}} \\ \boldsymbol{\sigma} \cdot \tilde{\mathbf{p}} & E-m \end{pmatrix} , \tag{170}$$

we obtain

$$\begin{pmatrix} E+m & \boldsymbol{\sigma} \cdot \tilde{\mathbf{p}} \\ \boldsymbol{\sigma} \cdot \tilde{\mathbf{p}} & E-m \end{pmatrix} \begin{pmatrix} \dot{\varphi}^{(n)} \\ \dot{\chi}^{(n)} \end{pmatrix} = \mathbf{0}, \quad \forall n = 0, 1, 2, 3, \ldots \,. \tag{171}$$

Thus for the lowest order case we can combine (171) for $n = 0$ with (168) to find a differential equation for $\varphi^{(0)}$, viz

$$\begin{aligned} \dot{\varphi}^{(0)} &= -\frac{\boldsymbol{\sigma} \cdot \tilde{\mathbf{p}}}{E+m} \dot{\chi}^{(0)} \\ &= -\frac{\boldsymbol{\sigma} \cdot \tilde{\mathbf{p}}}{E+m} \frac{d}{dt} \left(\frac{\boldsymbol{\sigma} \cdot \tilde{\mathbf{p}}}{E+m} \varphi^{(0)} \right) \end{aligned}$$

which leads to

$$\dot{\varphi}^{(0)} = -\frac{\boldsymbol{\sigma} \cdot \tilde{\mathbf{p}}}{2E} \frac{d}{dt} \left(\frac{\boldsymbol{\sigma} \cdot \tilde{\mathbf{p}}}{E+m} \right) \varphi^{(0)} \,, \tag{172}$$

where we use $\boldsymbol{\sigma} \cdot \tilde{\mathbf{p}} \, \boldsymbol{\sigma} \cdot \tilde{\mathbf{p}} = \tilde{\mathbf{p}}^2 = E^2 - m^2$. This further becomes

$$\dot{\varphi}^{(0)} = \left[-\frac{m\dot{E}}{2E(E+m)} - \frac{i\boldsymbol{\sigma} \cdot \tilde{\mathbf{p}} \times \dot{\tilde{\mathbf{p}}}}{2E(E+m)} \right] \varphi^{(0)} \,. \tag{173}$$

With regards to the first term we note that

$$\frac{d}{dt} \left(\sqrt{\frac{E+m}{2E}} \right) = -\frac{m\dot{E}}{2E(E+m)} \sqrt{\frac{E+m}{2E}} \,. \tag{174}$$

We then treat the second term as a time-dependent perturbation. The differential equation, due to the non-commutative nature of matrices, does not simply give the exponential solution, but rather the Taylor series expansion which can be rearranged to produce a time-ordered product. The result is known as an ordered (or path-ordered) exponential and we can thus write the spinor component as

$$\varphi^{(0)} = C\sqrt{\frac{E+m}{2E}}\, T\left(\exp\left[-i\int_0^t d\tau \frac{\boldsymbol{\sigma}\cdot\left(\tilde{\mathbf{p}}(\tau)\times\dot{\tilde{\mathbf{p}}}(\tau)\right)}{2E(\tau)(E(\tau)+m)}\right]\right) s\,, \tag{175}$$

where s is a spin eigenstate at $t=0$, chosen normalised, C a constant and T is the time-ordering operator. We note that the exponential notation is a short hand to represent the series expansion.

Defining

$$U(t) := T\left(\exp\left[-i\int_0^t d\tau \frac{\boldsymbol{\sigma}\cdot\left(\tilde{\mathbf{p}}(\tau)\times\dot{\tilde{\mathbf{p}}}(\tau)\right)}{2E(\tau)(E(\tau)+m)}\right]\right)\,, \tag{176}$$

we note that $U(t)$ is a unitary operator acting on s that can be considered as the time-evolution of the spin polarization. We define

$$\Lambda_p(t) = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}(t)\times\dot{\mathbf{p}}(t)}{(E_p(t)+m)}\,. \tag{177}$$

Note that $\Lambda_p(t)$ is Hermitian and traceless. There are two positive energy solutions. Thus s is one of the two spin (up or down) eigenstates. Define $s(t) = U(t)s$. The zeroth order term in the spinor expansion is thus (up to a multiplicative constant)

$$\begin{pmatrix} \varphi^{(0)} \\ \chi^{(0)} \end{pmatrix} = \sqrt{\frac{E+m}{2E}}\begin{pmatrix} s_\alpha(t) \\ \dfrac{\boldsymbol{\sigma}\cdot\tilde{\mathbf{p}}}{E+m} s_\alpha(t) \end{pmatrix}\,. \tag{178}$$

3.1.1. *First order correction.* We now look at the first order term in the spinor expansion i.e. the $\hbar$ correction term in the semiclassical expansion. We consequently return to the full-order positive spinor equation

$$\begin{pmatrix} E-m & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ -\boldsymbol{\sigma}\cdot\mathbf{p} & E+m \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} + i\hbar\begin{pmatrix} \dot{\varphi} \\ \dot{\chi} \end{pmatrix} = 0\,. \tag{179}$$

For ease of notation let

$$\Sigma := \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\,, \tag{180}$$

where $\mathbf{p}$ and E are time-dependent. Thus the zeroth order spinor term is

$$\begin{pmatrix} \varphi^{(0)} \\ \chi^{(0)} \end{pmatrix} = \sqrt{\frac{E+m}{2E}} \begin{pmatrix} s_\alpha(t) \\ \Sigma s_\alpha(t) \end{pmatrix} . \tag{181}$$

Recall that the spinors $s_\alpha(t)$ satisfy $s_\alpha^\dagger(t)s_\beta(t) = \delta_{\alpha\beta}$. Define the unitary matrix $S(t)$ as follows:

$$S(t) \equiv \sqrt{\frac{E+m}{2E}} \begin{pmatrix} U(t) & -\Sigma U(t) \\ \Sigma U(t) & U(t) \end{pmatrix} . \tag{182}$$

We note that

$$S^{-1}(t) \begin{pmatrix} m & \boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -m \end{pmatrix} S(t) = \begin{pmatrix} E & \mathbf{0} \\ \mathbf{0} & -E \end{pmatrix} . \tag{183}$$

Using this matrix, we change the representation of the spinors and let

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = S(t) \begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} . \tag{184}$$

The positive energy spinor equation (179) can be written in this representation as

$$\begin{pmatrix} \mathbf{0} \\ 2E\tilde{\chi} \end{pmatrix} + i\hbar S^{-1}(t)\frac{d}{dt}S(t) \begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} + i\hbar \begin{pmatrix} \dot{\tilde{\varphi}} \\ \dot{\tilde{\chi}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} . \tag{185}$$

We need to compute the matrix $S^{-1}(t)\dot{S}(t)$. Substituting the solution for $\varphi^{(0)}$ (175) back into Eq. (172) gives the relation

$$\frac{d}{dt}\left\{ \sqrt{\frac{E+m}{2E}} \begin{pmatrix} s_\alpha(t) \\ \Sigma s_\alpha(t) \end{pmatrix} \right\} = \left(\frac{E+m}{2E}\right)^{3/2} \begin{pmatrix} -\Sigma\dot{\Sigma} s_\alpha(t) \\ \dot{\Sigma} s_\alpha(t) \end{pmatrix} , \tag{186}$$

from which we also obtain

$$\frac{d}{dt}\left\{ \sqrt{\frac{E+m}{2E}} \begin{pmatrix} -\Sigma s_\alpha(t) \\ s_\alpha(t) \end{pmatrix} \right\} = -\left(\frac{E+m}{2E}\right)^{3/2} \begin{pmatrix} \dot{\Sigma} s_\alpha(t) \\ \Sigma\dot{\Sigma} s_\alpha(t) \end{pmatrix} . \tag{187}$$

Hence, if we define a 2×2 matrix $T(t)$ by

$$T(t) \equiv \frac{E+m}{2E} U^{\dagger}(t) \dot{\Sigma} U(t) \,, \tag{188}$$

then

$$S^{-1}(t) \frac{d}{dt} S(t) = \begin{pmatrix} \mathbf{0} & -T(t) \\ T(t) & \mathbf{0} \end{pmatrix} . \tag{189}$$

Substituting the matrix into Eq. (185) we obtain the two equations

$$\dot{\tilde{\varphi}} = T(t) \tilde{\chi} \,, \tag{190}$$

$$i\hbar \dot{\tilde{\chi}} = -i\hbar T(t) \tilde{\varphi} - 2E \tilde{\chi} \,. \tag{191}$$

Alternatively, using the semiclassical expansion (163) we obtain

$$\tilde{\chi}^{(n+1)} = -\frac{i}{2E} T(t) \tilde{\varphi}^{(n)} - \frac{i}{2E} \dot{\tilde{\chi}}^{(n)} \,, \tag{192}$$

$$\dot{\tilde{\varphi}}^{(n+1)} = T(t) \tilde{\chi}^{(n+1)} \,. \tag{193}$$

In this representation, the zeroth-order solutions are somewhat simpler: $\tilde{\chi}_{\alpha}^{(0)} = 0$ and $\tilde{\varphi}_1^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\tilde{\varphi}_2^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus,

$$\tilde{\chi}_{\alpha}^{(1)} = -\frac{i(E+m)}{4E^2} \begin{pmatrix} s_1^{\dagger}(t) \dot{\Sigma} s_{\alpha}(t) \\ s_2^{\dagger}(t) \dot{\Sigma} s_{\alpha}(t) \end{pmatrix} \tag{194}$$

and

$$\dot{\tilde{\varphi}}_{\alpha}^{(1)} = -\frac{i(E+m)^2}{8E^3} \begin{pmatrix} s_1^{\dagger}(t) \dot{\Sigma}^2 s_{\alpha}(t) \\ s_2^{\dagger}(t) \dot{\Sigma}^2 s_{\alpha}(t) \end{pmatrix} . \tag{195}$$

Now

$$\frac{d}{dt} \Sigma = \frac{1}{E+m} \left(\dot{\mathbf{p}} - \frac{\dot{E}}{E+m} \mathbf{p} \right) \cdot \boldsymbol{\sigma} \,. \tag{196}$$

Then we obtain

$$\left(\dot{\Sigma} \right)^2 = \frac{\dot{\mathbf{p}}^2 - \dot{E}^2}{(E+m)^2} = -\frac{\dot{p}^2}{(E+m)^2} \,, \tag{197}$$

where $\dot{p}^2 = \dot{p}^{\mu} \dot{p}_{\mu}$. Hence

$$\tilde{\varphi}_1^{(1)} = i g(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \tilde{\varphi}_2^{(1)} = i g(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{198}$$

where $g(t)$ is a real function defined by

$$g(t) \equiv \int_{t_0}^{t} \frac{\dot{p}(\tau)^2}{8E^3(\tau)}\, d\tau\,, \tag{199}$$

with t_0 being a constant. Changing back to the standard representation, the first-order spinor correction is

$$\begin{aligned}\hbar \begin{pmatrix} \varphi_\alpha^{(1)} \\ \chi_\alpha^{(1)} \end{pmatrix} &= \hbar S(t) \begin{pmatrix} \tilde{\varphi}_\alpha^{(1)} \\ \tilde{\chi}_\alpha^{(1)} \end{pmatrix} \\ &= i\hbar g(t) \begin{pmatrix} \varphi_\alpha^{(0)} \\ \chi_\alpha^{(0)} \end{pmatrix} - i\hbar \frac{(E+m)^{3/2}}{(2E)^{5/2}} \begin{pmatrix} -\Sigma\dot{\Sigma} s_\alpha(t) \\ \dot{\Sigma} s_\alpha(t) \end{pmatrix}.\end{aligned} \tag{200}$$

The semiclassical expansion for the positive energy spinor can now be written to order $\hbar$ as

$$\begin{aligned}\Phi(x) = C\sqrt{\frac{E+m}{2E}} &\left[(1 + i\hbar g(t)) \begin{pmatrix} s_\alpha(t) \\ \Sigma s_\alpha(t) \end{pmatrix} - i\hbar \frac{E+m}{(2E)^2} \begin{pmatrix} -\Sigma\dot{\Sigma} s_\alpha(t) \\ \dot{\Sigma} s_\alpha(t) \end{pmatrix} \right] \\ &\times \exp\left(-\frac{i}{\hbar} \int_0^t E(\xi) d\xi \right) e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}.\end{aligned} \tag{201}$$

Recall that the energy E and the matrix Σ are time dependent. However, the $\mathbf{p}$ in the exponential is not. We can choose the constant C to achieve the desired normalisation: $C = \sqrt{p_0/m}$. Rearranging, we obtain

$$\Phi(x) = \left[(1 + i\hbar g(t))\, u_\alpha^{(0)} - i\hbar \frac{E+m}{(2E)^2} \sqrt{\frac{E+m}{2m}} \begin{pmatrix} -\Sigma\dot{\Sigma} s_\alpha(t) \\ \dot{\Sigma} s_\alpha(t) \end{pmatrix} \right] \phi_{\mathbf{p}}(t) e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}, \tag{202}$$

where

$$u_\alpha^{(0)} = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} s_\alpha(t) \\ \Sigma s_\alpha(t) \end{pmatrix}, \tag{203}$$

can be considered the zeroth order spinor (for $V =$ constant it is the usual positive energy spinor), and

$$\phi_{\mathbf{p}}(t) = \sqrt{\frac{p_0}{E}} \exp\left(-\frac{i}{\hbar} \int_0^t E(\xi) d\xi \right), \tag{204}$$

is the time-dependent part of the WKB semiclassical expansion for the scalar field. We also define

$$u_\alpha^{(1)}(p) = i\hbar g(t)\sqrt{\frac{E+m}{2m}}\begin{pmatrix} s_\alpha \\ \Sigma s_\alpha \end{pmatrix} - i\hbar\frac{E+m}{(2E)^2}\sqrt{\frac{E+m}{2m}}\begin{pmatrix} -\Sigma\dot{\Sigma}s_\alpha \\ \dot{\Sigma}s_\alpha \end{pmatrix}, \quad (205)$$

as the first order spinor.

3.2. Negative Energy Solution. The Negative energy solutions are interpreted as the antiparticle solutions and thus this time we look for solutions of the form

$$\Psi(x) = \left[\begin{pmatrix} \varphi^{(0)} \\ \chi^{(0)} \end{pmatrix} + \hbar\begin{pmatrix} \varphi^{(1)} \\ \chi^{(1)} \end{pmatrix} + \hbar^2\begin{pmatrix} \varphi^{(2)} \\ \chi^{(2)} \end{pmatrix} + \ldots\right]\exp\left(+\frac{i}{\hbar}S\right)e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}. \quad (206)$$

The mode function satisfies the conjugate Dirac equation, and as the potential is the minimal substitution electromagnetic potential the result is that the antiparticle has opposite charge. Relative to the momentum operators we rewrite $\mathbf{V}(t) \to -\mathbf{V}(t)$. Due to the sign change, we obtain

$$i\hbar\begin{pmatrix} \dot{\varphi} \\ \dot{\chi} \end{pmatrix} - \begin{pmatrix} \varphi \\ \chi \end{pmatrix}\dot{S} + [\boldsymbol{\alpha}\cdot\tilde{\mathbf{p}}_+ - \beta m]\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0\,, \quad (207)$$

where this time we have $\tilde{\mathbf{p}}_+ = \mathbf{p} + \mathbf{V}(t)$. The lowest order equation gives the eigenvector equation with eigenvalue $\dot{S} = E_+$ where $E_+ = \sqrt{\tilde{\mathbf{p}}_+^2 + m^2}$ in keeping with the positive energy solutions. The spinor equation is now

$$i\hbar\begin{pmatrix} \dot{\varphi} \\ \dot{\chi} \end{pmatrix} - \begin{pmatrix} E_+ + m & -\boldsymbol{\sigma}\cdot\tilde{\mathbf{p}}_+ \\ -\boldsymbol{\sigma}\cdot\tilde{\mathbf{p}}_+ & E_+ - m \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0\,. \quad (208)$$

Comparing this equation with (179) for the positive energy solution we see that under the transformation $\hbar \to -\hbar$ and $\varphi \leftrightarrow \chi$ they are the same. Thus the negative energy solution can be written

$$\Psi(x) = \left[(1 - i\hbar g(t))\,v_\alpha^{(0)} + i\hbar\frac{E_+ + m}{(2E_+)^2}\sqrt{\frac{E_+ + m}{2m}}\begin{pmatrix} \dot{\Sigma}s_\alpha(t) \\ -\Sigma\dot{\Sigma}s_\alpha(t) \end{pmatrix}\right]$$
$$\times\,\bar{\phi}_{\mathbf{p}}^*(t)e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar}, \quad (209)$$

where

$$v_\alpha^{(0)} = \sqrt{\frac{E_+ + m}{2m}} \begin{pmatrix} \Sigma s_\alpha(t) \\ s_\alpha(t) \end{pmatrix} , \tag{210}$$

can be considered the zeroth order antiparticle spinor, and

$$\bar{\phi}^*_{\mathbf{p}}(t) = \sqrt{\frac{p_0}{E_+}} \exp\left(\frac{i}{\hbar} \int_0^t E_+(\xi) d\xi \right) , \tag{211}$$

is the time-dependent part of the WKB semiclassical expansion for the complex conjugate scalar field. Hence overall, the two solutions are related by $\hbar \to -\hbar$, $\varphi \leftrightarrow \chi$, $\mathbf{V}(t) \to -\mathbf{V}(t)$.

CHAPTER 3

Classical Position Shift

In this chapter we measure the effects radiation reaction in the classical theory of electrodynamics via the calculation of the position shift. We analyse the special case of linear acceleration before deriving a more general description.

1. Linear Acceleration

The case of linear acceleration simplifies matters considerably. Let us orientate our coordinate system such that the direction of the linear acceleration is along the z-axis. For the most part, we can consider the system to be in 1+1 dimensions (t, z). The reference frame can naturally be shifted so that the perpendicular velocities are zero. Before proceeding, we make a note of some simplifying notation in the spirit of Newton: We use dot notation to represent differentiation with respect to coordinate time t and dash notation to represent differentiation with respect to proper time τ. Thus

$$\dot{z} = \frac{dz}{dt} \qquad\qquad z' = \frac{dz}{d\tau}\,. \tag{212}$$

To enable the reader to easily follow the calculations and indeed for ease of reproducing them, we give a number of simple identities which are of use in this system. Firstly, in 1+1 dimensions the relativistic gamma factor is defined, in our above notation, as

$$t' = \gamma = \frac{1}{\sqrt{1-\dot{z}^2}}\,. \tag{213}$$

The following are equalities between the dot and dash representations

$$\dot{\gamma} = \gamma^3 \ddot{z}\dot{z}\,, \tag{214}$$

$$t'' = \gamma' = \gamma^4 \ddot{z}\dot{z}\,, \tag{215}$$

$$z' = \gamma \dot{z}\,, \tag{216}$$

$$z'' = \gamma^4 \ddot{z}\,, \tag{217}$$

$$z''' = \gamma^5 \dddot{z} + 4\gamma^7 \ddot{z}^2 \dot{z}\,. \tag{218}$$

Recall that the Lorentz-Dirac force is given by

$$F^\mu_{\rm LD} \equiv \frac{2\alpha_c}{3}\left[\frac{d^3x^\mu}{d\tau^3} + \frac{dx^\mu}{d\tau}\left(\frac{d^2x^\nu}{d\tau^2}\frac{d^2x_\nu}{d\tau^2}\right)\right]\,. \tag{26}$$

In this system the expression for the force can be much simplified. For the z component one finds

$$\begin{aligned}
F^z_{\rm LD} &= \frac{2\alpha_c}{3}\left[z''' + z'\left[(t'')^2 - (z'')^2\right]\right] \\
&= \frac{2\alpha_c}{3}\left[\gamma^5\dddot{z} + 4\gamma^7\ddot{z}^2\dot{z} + \gamma\dot{z}\left[\gamma^8\ddot{z}^2\dot{z}^2 - \gamma^8\ddot{z}^2\right]\right] \\
&= \frac{2\alpha_c}{3}\left[\gamma^5\dddot{z} + 4\gamma^7\ddot{z}^2\dot{z} - \gamma^7\ddot{z}^2\dot{z}\right] \\
&= \frac{2\alpha_c}{3}\left[\gamma^5\dddot{z} + 3\gamma^7\ddot{z}^2\dot{z}\right] \\
&= \frac{2\alpha_c}{3}\gamma^2\left[\gamma^3\dddot{z} + 3\gamma^5\ddot{z}^2\dot{z}\right] \\
&= \frac{2\alpha_c}{3}\gamma^2 d_t\left(\gamma^3\ddot{z}\right)\,.
\end{aligned} \tag{219}$$

The t component can similarly be given as

$$F^t_{\rm LD} = \frac{2\alpha_c}{3}\dot{z}\gamma^2 d_t\left(\gamma^3\ddot{z}\right)\,. \tag{220}$$

Now, for linear acceleration in the potential $V(z)$, the external force acting on the particle is given by

$$\begin{aligned}
F^t_{\rm ext} &= -V'(z)\,dz/d\tau\,, \\
F^z_{\rm ext} &= -V'(z)\,dt/d\tau\,, \\
F^x_{\rm ext} &= F^y_{\rm ext} = 0\,.
\end{aligned} \tag{221}$$

The Lorentz-Dirac force can be similarly written as

$$
\begin{aligned}
F^t_{\mathrm{LD}} &= F_{\mathrm{LD}}\, dz/d\tau\,, \\
F^z_{\mathrm{LD}} &= F_{\mathrm{LD}}\, dt/d\tau\,, \\
F^x_{\mathrm{LD}} &= F^y_{\mathrm{LD}} = 0
\end{aligned}
\tag{222}
$$

where, using the more compact form found above, we have

$$
F_{\mathrm{LD}} \equiv \frac{2\alpha_c}{3}\gamma\frac{d}{dt}(\gamma^3\ddot{z})\,. \tag{223}
$$

1.1. Space-dependent Potential. In this section we explicitly calculate the position shift for linear acceleration due to the potential $V(z)$, where z is the direction of the acceleration. We recall that the position shift is the change in position due to radiation reaction. We also recall, that we shall regard the radiation reaction force as a perturbation. What this means in practice is that all quantities, such as $\dot{z}$, $\ddot{z}$ and $\dddot{z}$, in the equations involving the radiation reaction force are evaluated using the original unperturbed path given by $ma^\mu = F^\mu_{\mathrm{ext}}$. We shall find the position shift to first non-trivial order in F_{LD}.

Suppose that, in the absence of radiation reaction, the particle would be at $z = 0$ at time $t = 0$. This is the position of the unperturbed particle obeying $ma^\mu = F^\mu_{\mathrm{ext}}$. The position of the particle undergoing radiation reaction, and thus obeying $ma^\mu = F^\mu_{\mathrm{ext}} + F^\mu_{\mathrm{LD}}$, is equal the position shift, which we label δz. In the system with $V(z)$, the calculation of δz is facilitated by the observation that the change in the total energy, $m\, dt/d\tau + V(z)$, is equal to the work done by the Lorentz-Dirac force. We then find

$$
\begin{aligned}
\int_{-\infty}^{t} F_{\mathrm{LD}}(t')\dot{z}(t')dt' &= \frac{d}{d\dot{z}}\frac{m}{\sqrt{1-\dot{z}^2}}\delta\dot{z} + V'(z)\delta z \\
&= m\gamma^3\dot{z}^2\frac{d}{dt}\left(\frac{\delta z}{\dot{z}}\right)\,,
\end{aligned}
\tag{224}
$$

where we have used

$$
\frac{d}{dt}\left(m\gamma\dot{z}\right) = m\gamma^3\ddot{z} = -V'(z)\,. \tag{225}
$$

This last line is to zeroth-order in $F_{\rm LD}$ as we explained above. Rearranging and integrating, we obtain the position shift:

$$\delta z_{\rm LD} = \frac{v_0}{m}\int_{-\infty}^{0}\left(\int_{-\infty}^{t} F_{\rm LD}\frac{dz}{dt'}dt'\right)\frac{1}{\gamma^3(t)[\dot{z}(t)]^2}dt\,, \tag{226}$$

where $v_0 = \dot{z}(0)$ is the final velocity. The reader may note that the outer integration limit is $t = 0$, as is the time of measurement for v_0, which is of course due to the fact that $\delta z_{\rm LD}$ is the position shift at $t = 0$.

Now, the current set-up, in which the unperturbed particle is at the origin at the time of measurement, is naturally made for simplicity and we indeed have complete freedom to do so by appropriate definition of the coordinate system. However, it does encourage the question as to what the position shift would be if this were not the case, i.e. if $z = z_0 \neq 0$ at $t = 0$ as opposed to $z = 0$. We assume that z_0 is still in the final non-accelerated region and thus the final velocity is still v_0. The result is that the time the particle spends between the end of the acceleration and the measurement at $t = 0$ is lengthened by $t_0 = z_0/v_0$. The effect is the same as shifting the entire trajectory earlier in time by t_0. Consequently, we may calculate the new position shift by using our original trajectory and taking the measurement at $t = t_0$ instead of $t = 0$. The extra contribution to the position shift is thus

$$\delta z_{\rm extra} = \frac{v_0}{m}\int_{0}^{t_0}\left(\int_{-\infty}^{t} F_{\rm LD}\frac{dz}{dt'}dt'\right)\frac{1}{\gamma^3(t)[\dot{z}(t)]^2}dt\,, \tag{227}$$

which is easily obtained with reference to the earlier comments about the limits and the constant velocity. Within the new limits, $t \in [0, t_0]$, we note that $\dot{z}(t) = v_0$ and $\gamma(t) = \gamma_0 \equiv (1 - v_0^2)^{-1/2}$ are constant and $F_{\rm LD} = 0$. We can interchange the order of integration to find

$$\begin{aligned}\delta z_{\rm extra} &= \frac{v_0}{m}\int_{-\infty}^{0}\left(\int_{0}^{t_0}\frac{1}{\gamma_0^3 v_0^2}dt'\right)F_{\rm LD}\frac{dz}{dt}dt \\ &= -\frac{z_0}{m\gamma_0^3 v_0^2}E_{\rm em}\,,\end{aligned} \tag{228}$$

where E_{em} is the energy emitted as radiation given by

$$\begin{aligned} E_{\mathrm{em}} &= -\int_{-\infty}^{0} F_{\mathrm{LD}} \frac{dz}{dt} dt \\ &= \frac{2\alpha_c}{3} \int_{-\infty}^{0} (\gamma^3 \ddot{z})^2 \, dt \,. \end{aligned} \tag{229}$$

This is the relativistic Larmor formula for one-dimensional motion.

The current form of the position shift (226), whilst useful for the above comment, is somewhat more complicated than is necessary. After interchanging the order of integration to obtain

$$\delta z_{\mathrm{LD}} = -\frac{v_0}{m} \int_{-\infty}^{0} \left(\int_0^t \frac{1}{\gamma^3(t')[\dot{z}(t')]^2} dt' \right) F_{\mathrm{LD}} \frac{dz}{dt} dt \,, \tag{230}$$

it can be simplified by noting that, for the space dependent potential,

$$\left(\frac{\partial z}{\partial p} \right)_t = \frac{v_0}{m} \dot{z}(t) \int_0^t \frac{1}{\gamma^3(t') \left[\dot{z}(t') \right]^2} \, dt' \,, \tag{231}$$

where p is the final momentum of the particle. This equation can be demonstrated as follows. Since the energy is conserved, we have

$$\begin{aligned} \sqrt{p^2 + m^2} &= \sqrt{(mz')^2 + m^2} + V(z) \\ &= \frac{m}{\sqrt{1 - \dot{z}^2}} + V(z) \,, \end{aligned} \tag{232}$$

and hence,

$$\dot{z} = \left[1 - m^2 \left(\sqrt{p^2 + m^2} - V(z) \right)^{-2} \right]^{1/2} . \tag{233}$$

By differentiating both sides with respect to p with t fixed, and noting that

$$p/\sqrt{p^2 + m^2} = v_0 \,, \tag{234}$$

$$\sqrt{p^2 + m^2} - V(z) = m\gamma \,, \tag{235}$$

we obtain

$$\frac{d}{dt} \left(\frac{\partial z}{\partial p} \right)_t = \frac{1}{m\gamma^3 \dot{z}} \left[v_0 - V'(z) \left(\frac{\partial z}{\partial p} \right)_t \right] . \tag{236}$$

By substituting the formula $V'(z) = -m\gamma^3 \ddot{z}$ (see (225)) in (236) we find

$$\frac{d}{dt} \left[\frac{1}{\dot{z}} \left(\frac{\partial z}{\partial p} \right)_t \right] = \frac{v_0}{m\gamma^3 \dot{z}^2} \,. \tag{237}$$

Then by integrating this formula, remembering that $z = 0$ at $t = 0$ for all p, we arrive at (231). Consequently, the position shift can be written in the more compact form

$$\delta z_{\mathrm{LD}} = -\int_{-\infty}^{0} dt\, F_{\mathrm{LD}} \left(\frac{\partial z}{\partial p}\right)_t . \tag{238}$$

1.2. Time-dependent Potential. The case of linear acceleration due to a time-dependent potential can be analysed in a similar way to the previous exercise. We again define the coordinate system such that the acceleration is in the direction of the z-axis, but this time the potential is given by $V(t)$. In the $V(z)$ case, we use the energy conservation, whereas now we shall make use of the momentum conservation equation which reads

$$\frac{d}{dt}\left[m\gamma\dot{z} + V(t)\right] = F_{\mathrm{LD}} . \tag{239}$$

The lack of symmetry between the two situations ($V(z)$ and $V(t)$) is worth noting. In both situations we are measuring the change in position at equal time, as opposed to the possible consideration of the change in time for the same position. Retaining the same measurement breaks some of the symmetry. Returning to the change in momentum, we note that in the time-dependent case, the potential *at equal time* is the same for the particle in the presence or absence of radiation reaction. The momentum conservation (239) thus leads to

$$\delta(m\gamma\dot{z}) = m\gamma^3 \frac{d}{dt}(\delta z) = \int_{-\infty}^{t} F_{\mathrm{LD}}(t')\, dt' . \tag{240}$$

Rearranging for δz and interchanging the order of integration as per the previous case, the position shift is given by

$$\delta z_{\mathrm{LD}} = -\int_{-\infty}^{0} \left(\int_0^t \frac{1}{m\gamma^3} dt'\right) F_{\mathrm{LD}} dt . \tag{241}$$

In line with the spatially dependent potential case, this can be further simplified. In fact, we find that for the t-dependent potential,

$$\left(\frac{\partial z}{\partial p}\right)_t = \int_0^t \frac{1}{m\gamma^3}\, dt , \tag{242}$$

where we again recall that this is for the unperturbed particle. This is demonstrated as follows: The momentum conservation for this particle in the z-direction reads

$$m\frac{dz}{d\tau} + V(t) = p\,. \tag{243}$$

Hence, with the condition $z = 0$ at $t = 0$, we find

$$z = \int_0^t \left\{1 + \frac{m^2}{[p - V(t)]^2}\right\}^{-1/2} dt\,. \tag{244}$$

By differentiating this expression with respect to p and using $p - V(t) = m\,dz/d\tau$ and $\sqrt{[p - V(t)]^2 + m^2} = m\,dt/d\tau$, we indeed obtain (242).

Consequently, we note that the position shift can be written in the same form as (238) before, namely

$$\delta z_{\mathrm{LD}} = -\int_{-\infty}^{0} dt\, F_{\mathrm{LD}} \left(\frac{\partial z}{\partial p}\right)_t\,. \tag{245}$$

2. Generalised Classical Position Shift

The fact that both the t-dependent and z-dependent potentials for linear acceleration lead ultimately to the same expression for the classical position shift and in addition that this expression is fairly simple, leads one to suspect a more general argument for this formula. This is indeed the case as we now proceed to relate.

We now look at the full three spatial dimensional system. The system is one where the total force acting on the particle is the sum of an external force F and an additional force $\gamma\mathcal{F}$, which we intend to treat as a perturbation:

$$m\frac{d^2x^i}{d\tau^2} = F^i + \mathcal{F}^i\frac{dt}{d\tau}\,. \tag{246}$$

2.1. Homogeneous system. As yet, we have said nothing about what these forces are. Let us consider this system to be the result of a perturbation from a Hamiltonian system i.e. that in the absence of the extra force, $\mathcal{F} = 0$, the system is described by a Hamiltonian $H(\mathbf{x}, \mathbf{p})$, where $(\mathbf{x}, \mathbf{p})$ are the

generalised coordinates and conjugate momenta. We shall refer to this system as the homogeneous system. Hamilton's equations are given by

$$\dot{x^i} = \frac{\partial H}{\partial p^i}, \tag{247}$$

$$\dot{p^i} = -\frac{\partial H}{\partial x^i}. \tag{248}$$

Consider a perturbation to the solution $(\mathbf{x}, \mathbf{p})$ given by $(\mathbf{x} + \mathbf{\Delta x}, \mathbf{p} + \mathbf{\Delta p})$. We shall refer to these perturbations to the path as the homogeneous perturbations. The expansion to second order of the Hamiltonian of the perturbed solution is given by

$$H(\mathbf{x} + \mathbf{\Delta x}, \mathbf{p} + \mathbf{\Delta p}) = H(\mathbf{x}, \mathbf{p}) + \frac{\partial H}{\partial x^i}\Delta x^i + \frac{\partial H}{\partial p^i}\Delta p^i + \frac{1}{2}\left[\frac{\partial^2 H}{\partial x^i \partial x^j}\Delta x^i \Delta x^j + 2\frac{\partial^2 H}{\partial x^i \partial p^j}\Delta x^i \Delta p^j + \frac{\partial^2 H}{\partial p^i \partial p^j}\Delta p^i \Delta p^j\right]. \tag{249}$$

The equations for the homogeneous perturbations are given by

$$\Delta \dot{x^i} = \Delta \frac{\partial H}{\partial p^i} = \frac{\partial^2 H}{\partial x^j \partial p^i}\Delta x^j + \frac{\partial^2 H}{\partial p^j \partial p^i}\Delta p^j, \tag{250}$$

$$\Delta \dot{p^i} = -\Delta \frac{\partial H}{\partial x^i} = -\frac{\partial^2 H}{\partial x^j \partial x^i}\Delta x^j - \frac{\partial^2 H}{\partial p^j \partial x^i}\Delta p^j. \tag{251}$$

Thus these can be seen to be generated by the equations

$$\Delta \dot{x^i} = \frac{\partial \bar{H}}{\partial \Delta p^i}, \tag{252}$$

$$\Delta \dot{p^i} = -\frac{\partial \bar{H}}{\partial \Delta x^i}, \tag{253}$$

where the Hamiltonian $\bar{H}$ is given by the second order terms in the expansion in (249), viz

$$\bar{H} = \frac{1}{2}\left[\frac{\partial^2 H}{\partial x^i \partial x^j}\Delta x^i \Delta x^j + 2\frac{\partial^2 H}{\partial x^i \partial p^j}\Delta x^i \Delta p^j + \frac{\partial^2 H}{\partial p^i \partial p^j}\Delta p^i \Delta p^j\right]. \tag{254}$$

This can be rewritten in terms of the matrices A_{ij}, B_{ij} and C_{ij}, where A and C are symmetric, as follows

$$\bar{H} = \frac{1}{2}\left[A_{ij}\Delta p^i \Delta p^j + 2B_{ij}\Delta x^i \Delta p^j + C_{ij}\Delta x^i \Delta x^j\right] . \tag{255}$$

Thus the equations (252) and (253) can be written

$$\Delta \dot{x^i} = \frac{\partial \bar{H}}{\partial \Delta p^i} = B_{ji}\Delta x^j + C_{ij}\Delta p^j , \tag{256}$$

$$\Delta \dot{p^i} = -\frac{\partial \bar{H}}{\partial \Delta x^i} = -A_{ij}\Delta x^j - B_{ij}\Delta p^j . \tag{257}$$

As a consequence we can deduce that the symplectic product of the perturbations is conserved. Given two solutions $(\Delta X^i, \Delta P^i)$, $(\Delta x^i, \Delta p^i)$, the symplectic product is given by

$$\langle \Delta X^i, \Delta P^i | \Delta x^i, \Delta p^i \rangle = \Delta X^i \Delta p^i - \Delta x^i \Delta P^i . \tag{258}$$

The time conservation is easily seen as follows:

$$\begin{aligned}
&\frac{d}{dt}\left(\Delta X^i \Delta p^i - \Delta x^i \Delta P^i\right)\\
&= \Delta \dot{X^i} \Delta p^i + \Delta X^i \Delta \dot{p^i} - \Delta \dot{x^i} \Delta P^i - \Delta x^i \Delta \dot{P^i}\\
&= \left(B_{ji}\Delta X^j + C_{ij}\Delta P^j\right)\Delta p^i + \left(-A_{ij}\Delta x^j - B_{ij}\Delta p^j\right)\Delta X^i\\
&\quad - \left(B_{ji}\Delta x^j + C_{ij}\Delta p^j\right)\Delta P^i - \left(-A_{ij}\Delta X^j - B_{ij}\Delta P^j\right)\Delta x^i\\
&= \Delta X^j \Delta p^i \left(B_{ji} - B_{ji}\right) + \Delta x^j \Delta P^i \left(-B_{ji} + B_{ij}\right)\\
&\quad + \Delta P^j \Delta p^i \left(C_{ij} - C_{ji}\right) + \Delta x^j \Delta X^i \left(-A_{ij} + A_{ji}\right)\\
&= 0 ,
\end{aligned} \tag{259}$$

where we have made use of the fact that A_{ij} and C_{ij} are symmetric.

2.2. Inhomogeneous system. We now consider the system for which we add the additional force, $\mathcal{F} \neq 0$. We shall refer to this system as the inhomogeneous system. Let $(\mathbf{x}, \mathbf{p}) = (\mathbf{x}_0(t), \mathbf{p}_0(t))$ be a solution to the homogeneous system such that $(\mathbf{x}_0(0), \mathbf{p}_0(0)) = (0, \mathbf{p})$. This solution gives the classical trajectory of a particle passing through the origin at $t = 0$ with momentum $\mathbf{p}$ in

the absence of radiation reaction. We let $(\mathbf{x}_0(t)+\delta\mathbf{x}(t), \mathbf{p}_0(t)+\delta\mathbf{p}(t))$ be a solution to the inhomogeneous system to first order in $\mathcal{F}$. The $(\delta\mathbf{x}(t), \delta\mathbf{p}(t))$, which we call the inhomogeneous perturbations, are the perturbations to the classical trajectory due to the addition of the radiation reaction force, treated as a perturbation to first order. They have the property that $((\delta\mathbf{x}(t), \delta\mathbf{p}(t)) \to (0,0)$ as $t \to -\infty$ and will satisfy the equations

$$\begin{aligned} \frac{d}{dt}\delta x^i &= B_{ji}\delta x^j + C_{ij}\delta p^j \,, \\ \frac{d}{dt}\delta p^i &= -A_{ij}\delta x^j - B_{ij}\delta p^j + \mathcal{F} \,. \end{aligned} \tag{260}$$

In order to solve these equations, we define a set of homogeneous perturbations $(\Delta\mathbf{x}_{(j)}(t;s), \Delta\mathbf{p}_{(j)}(t;s))$, with $j = 1,2,3$ and $s \in (-\infty,\infty)$, by the following initial conditions:

$$\begin{aligned} \Delta x^i_{(j)}(s;s) &= 0 \,, \\ \Delta p^i_{(j)}(s;s) &= \delta^i_j \,. \end{aligned} \tag{261}$$

The solution $(\mathbf{x}_0 + \Delta\mathbf{x}_{(j)}(t;s), \mathbf{p}_0 + \Delta\mathbf{p}_{(j)}(t;s))$ then represents the particle trajectory which coincides with $\mathbf{x}_0(t)$ at time $t = s$, but which has excess momentum solely in the j-direction at this time. This trajectory is represented in Fig. 3.1. With these solutions now defined we note that the solutions of the coupled inhomogeneous equations can be given by

$$\begin{aligned} \delta x^i &= \int_{-\infty}^{t} ds\mathcal{F}^j(s)\Delta x^i_{(j)}(t;s) \,, \\ \delta p^i &= \int_{-\infty}^{t} ds\mathcal{F}^j(s)\Delta p^i_{(j)}(t;s) \,. \end{aligned} \tag{262}$$

where the index j is summed over. The position shift due to the additional force $\mathcal{F}$ can therefore be written as

$$\delta x^i = \int_{-\infty}^{0} dt\mathcal{F}^j(t)\Delta x^i_{(j)}(0;t) \,. \tag{263}$$

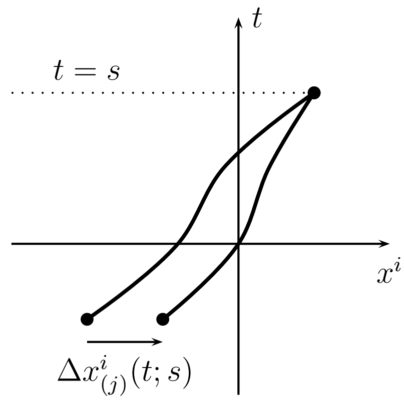

FIGURE 3.1. The world lines for the solutions $(\mathbf{x}_0(t), \mathbf{p}_0(t))$, which passes through the origin, and $(\mathbf{x}_0 + \Delta\mathbf{x}_{(j)}(t; s), \mathbf{p}_0 + \Delta\mathbf{p}_{(j)}(t; s))$ for some j.

With final momentum at time $t = 0$ as p then, given the definition of the $\Delta x^i_{(j)}(t; s)$ solutions, we can write

$$\left(\frac{\partial x^j}{\partial p^i}\right)_t = \frac{\Delta x^j_{(i)}(t; 0)}{\Delta p^i_{(i)}(0; 0)} = \Delta x^j_{(i)}(t; 0)\,. \tag{264}$$

Recall that the symplectic product of homogeneous perturbations is conserved. Thus, by equating the symplectic products of the two solutions $\big(\Delta\mathbf{x}_{(i)}(t; s), \Delta p_{(i)}(t; s)\big)$ and $\big(\Delta\mathbf{x}_{(j)}(t; u), \Delta p_{(j)}(t; u)\big)$ at the times $t = s$ and $t = u$ we have

$$\begin{aligned}\Delta\mathbf{x}_{(i)}(s; s) \cdot \Delta\mathbf{p}_{(j)}(s; u) - \Delta\mathbf{x}_{(j)}(s; u) \cdot \Delta\mathbf{p}_{(i)}(s; s)&\\ = \Delta\mathbf{x}_{(i)}(u; s)\Delta\mathbf{p}_{(j)}(u; u) - \Delta\mathbf{x}_{(j)}(u; u)\Delta\mathbf{p}_{(i)}(u; s)\,.&\end{aligned} \tag{265}$$

This equation and the initial conditions that define these solutions imply

$$\Delta x^i_{(j)}(s; u) = -\Delta x^j_{(i)}(u; s)\,. \tag{266}$$

Of particular interest to us, we obtain $\Delta x^i_{(j)}(0; t) = -\Delta x^j_{(i)}(t; 0)$, which means that we can rewrite the position shift as

$$\delta x^i = -\int_{-\infty}^{0} dt \mathcal{F}^j \left(\frac{\partial x^j}{\partial p^i}\right)_t\,. \tag{267}$$

This is the same form of equation which we derived for the linear acceleration and the Lorentz-Dirac force to first order. In that case, we only have non-zero

terms for $j = 3$ with $\mathcal{F}^z = \mathcal{F}^z_{\mathrm{LD}}$ and $i = 3$. We have thus derived a more general relation for the classical position shift. We have assumed that the position shift is due to an additional force taken as a perturbation, to first order, to a Hamiltonian system.

The above conditions apply for the specific model we wish to consider for the Lorentz-Dirac force in three dimensional motion. Here we simply write $\mathcal{F}^i = \mathcal{F}^i_{\mathrm{LD}} \equiv \gamma^{-1} F^i_{\mathrm{LD}}$. Thus

$$m\frac{d^2x^i}{d\tau^2} = F^i + \mathcal{F}^i_{\mathrm{LD}}\frac{dt}{d\tau}\,. \tag{268}$$

It would be useful to repeat here for the 3D case the conversion from proper time variables to coordinate time variables as was done for the 1D case at the beginning of the chapter. Using the dot notation for differentiation with respect to t, as before, we define $\mathbf{v} = \dot{\mathbf{x}}$ and $\mathbf{a} = \ddot{x}$ and note that

$$\gamma = \frac{dt}{d\tau} = \frac{1}{(1 - \mathbf{v}\cdot\mathbf{v})^{\mathbf{1/2}}} \tag{269}$$

$$\dot{\gamma} = \frac{\mathbf{v}\cdot\mathbf{a}}{(1-\mathbf{v}^2)^{3/2}} = \gamma^3\mathbf{v}\cdot\mathbf{a}\,, \tag{270}$$

and also

$$\frac{dx^i}{d\tau} = \frac{dt}{d\tau}\frac{dx^i}{dt} = \gamma v^i \tag{271}$$

$$\begin{aligned}\frac{d^2x^i}{d\tau^2} &= \gamma\frac{d}{dt}\left(\gamma v^i\right)\\ &= \gamma\left(\dot{\gamma}v^i + \gamma a^i\right)\\ &= \gamma^4\mathbf{v}\cdot\mathbf{a}v^i + \gamma^2 a^i\end{aligned} \tag{272}$$

$$\frac{d^2t}{d\tau^2} = \gamma\dot{\gamma} = \gamma^4\mathbf{v}\cdot\mathbf{a}\,. \tag{273}$$

From these last two relations, we have

$$\begin{aligned}\frac{d^2x^\nu}{d\tau^2}\frac{d^2x_\nu}{d\tau^2} &= \left(\gamma^4\mathbf{v}\cdot\mathbf{a}\right)^2 - \left(\gamma^4\mathbf{v}\cdot\mathbf{a}\mathbf{v}+\gamma^2\mathbf{a}\right)\cdot\left(\gamma^4\mathbf{v}\cdot\mathbf{a}\mathbf{v}+\gamma^2\mathbf{a}\right)\\ &= \gamma^8\left(\mathbf{v}\cdot\mathbf{a}\right)^2 - \gamma^8\left(\mathbf{v}\cdot\mathbf{a}\right)^2\mathbf{v}^2 - 2\gamma^6\left(\mathbf{v}\cdot\mathbf{a}\right)^2 - \gamma^4\mathbf{a}^2\\ &= \gamma^8\left(\mathbf{v}\cdot\mathbf{a}\right)^2\gamma^{-2} - 2\gamma^6\left(\mathbf{v}\cdot\mathbf{a}\right)^2 - \gamma^4\mathbf{a}^2\\ &= \gamma^6\left(\mathbf{v}\cdot\mathbf{a}\right)^2 - \gamma^4\mathbf{a}^2\,. \end{aligned} \tag{274}$$

Now, the Lorentz-Dirac force is given by

$$F^\mu_{\mathrm{LD}} \equiv \frac{2\alpha_c}{3}\left[\frac{d^3x^\mu}{d\tau^3} + \frac{dx^\mu}{d\tau}\left(\frac{d^2x^\nu}{d\tau^2}\frac{d^2x_\nu}{d\tau^2}\right)\right]\,. \tag{26}$$

Thus the spatial components can be written

$$F^i_{\mathrm{LD}} = \frac{2\alpha_c}{3}\left[\gamma\frac{d}{dt}\left(\frac{d^2x^i}{d\tau^2}\right) + \gamma v^i\left(\frac{d^2x^\nu}{d\tau^2}\frac{d^2x_\nu}{d\tau^2}\right)\right]\,. \tag{275}$$

Substituting the relations above, we have

$$F^i_{\mathrm{LD}} = \frac{2\alpha_c}{3}\gamma\left[\frac{d}{dt}\left(\gamma^4\mathbf{v}\cdot\mathbf{a}v^i + \gamma^2 a^i\right) + v^i\left(\gamma^6\left(\mathbf{v}\cdot\mathbf{a}\right)^2 - \gamma^4\mathbf{a}^2\right)\right]\,, \tag{276}$$

and consequently,

$$\mathcal{F}^i_{\mathrm{LD}} = \frac{2\alpha_c}{3}\left[\frac{d}{dt}\left(\gamma^4\mathbf{v}\cdot\mathbf{a}v^i + \gamma^2 a^i\right) + v^i\left(\gamma^6\left(\mathbf{v}\cdot\mathbf{a}\right)^2 - \gamma^4\mathbf{a}^2\right)\right]\,. \tag{277}$$

It will be sufficient, and more useful, for our purposes to leave this expression in its current form instead of rearranging or evaluating it further.

Returning to the equations of motion (268), the external force on the charged particle F, which causes the initial acceleration in the first place, merely needs to be one derived from a Hamiltonian, which is the case for most external forces that we would consider. For example, the most natural external force on a charged particle would be an electromagnetic Lorentz force. The Lorentz force can be derived from the Hamiltonian

$$H = \sqrt{\left(\mathbf{p} - e\mathbf{A}_{\mathrm{ext}}\right)^2 + m^2} + eA^0_{\mathrm{ext}}\,, \tag{278}$$

where A^μ_{ext} is the external electromagnetic potential. In our previous notation, the potential is $V^\mu = eA^\mu_{\mathrm{ext}}$. Henceforth we refer to

$$\delta x^i_C = -\int_{-\infty}^{0} dt \mathcal{F}_{\mathrm{LD}}{}^j \left(\frac{\partial x^j}{\partial p^i}\right)_t . \tag{279}$$

as the classical position shift (due to electromagnetic radiation reaction as described by the Lorentz-Dirac force to first order). This concludes our investigation using the classical theory and provides us with the classical position shift, with which we can compare the results of investigations using quantum field theory.

CHAPTER 4

Scalar Quantum Position Shift

In this chapter we derive the contributions to the position shift in the quantum scalar electrodynamics model. We calculate the contributions from the photon emission, forward scattering and renormalisation counterterm perturbation effects and combine them to compare the position shift in the $\hbar \to 0$ limit with that from the classical theory.

1. Initial control state

The initial state is given by the incoming wave packet in (74) as

$$|i\rangle = \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} f(\mathbf{p}) A^\dagger(\mathbf{p})|0\rangle \,, \tag{280}$$

with $f(\mathbf{p})$ peaked about the initial momentum in the region $\mathcal{M}_-$. Let the potential satisfy $|V_0| < 2m$, thus precluding the possibility of scalar-particle pair creation. This would be a vacuum process and thus not of interest to us in examining the evolution of the particle under consideration. The 'free' particle that we wish to use as our control measurement does not interact with the electromagnetic field via radiation reaction. Having passed through the classical non-perturbative potential V in the region $\mathcal{M}_I$, the final state can be considered as analogous to the initial state, albeit with the wave packet peaked about the final momentum in the region $\mathcal{M}_+$. Thus, we wish to find the position expectation value for the particle in the above state $|i\rangle$. Under the above restriction to the potential, if there is only one particle in the state, then the probability density coincides with the charge density $\rho(x)$ given in

(69). The position expectation value is then given by

$$\langle \mathbf{x} \rangle = \int d^3\mathbf{x}\, \mathbf{x} \langle \rho(t, \mathbf{x}') \rangle \,. \tag{281}$$

The expectation value of the charge, and thus probability, density for the initial state $|i\rangle$ is as follows

$$\begin{aligned}
\langle i|\rho(t,\mathbf{x})|i\rangle &= \langle i|\frac{i}{\hbar} : \varphi^\dagger \overleftrightarrow{\partial_t} \varphi : |i\rangle \\
&= \int \frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3} \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} \langle 0| \\
&\times f^*(\mathbf{p}')A(\mathbf{p}')\frac{i}{\hbar} : \varphi^\dagger \overleftrightarrow{\partial_t} \varphi : f(\mathbf{p})A^\dagger(\mathbf{p})|0\rangle \\
&= \frac{i}{\hbar}\hbar^2 \int \frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3} \frac{d^3\mathbf{p}''}{2p_0''(2\pi\hbar)^3} \frac{d^3\mathbf{p}'''}{2p_0'''(2\pi\hbar)^3} \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} \\
&\times \langle 0|f^*(\mathbf{p}')A(\mathbf{p}') \Big[A^\dagger(\mathbf{p}'')\Phi^\dagger_{\mathbf{p}''}(\mathbf{x},t)A(\mathbf{p}''')\partial_t\Phi_{\mathbf{p}'''}(\mathbf{x},t) \\
&-A^\dagger(\mathbf{p}'')\partial_t\Phi^\dagger_{\mathbf{p}''}(\mathbf{x},t)A(\mathbf{p}''')\Phi_{\mathbf{p}'''}(\mathbf{x},t)\Big] f(\mathbf{p})A^\dagger(\mathbf{p})|0\rangle \,,
\end{aligned} \tag{282}$$

where we have made use of the lack of the pair creation to remove the $B(\mathbf{p})$ antiparticle creation/annihilation operators. Proceeding, using the commutation relations set out in (73):

$$\begin{aligned}
\langle i|\rho(t,\mathbf{x})|i\rangle &= i\hbar \int \frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3} \frac{d^3\mathbf{p}''}{2p_0''(2\pi\hbar)^3} \frac{d^3\mathbf{p}'''}{2p_0'''(2\pi\hbar)^3} \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} \\
&\times f^*(\mathbf{p}')f(\mathbf{p})2p_0''(2\pi\hbar)^3\delta(\mathbf{p}'-\mathbf{p}'')2p_0'''(2\pi\hbar)^3\delta(\mathbf{p}'''-\mathbf{p}) \\
&\times [\Phi_{\mathbf{p}''}(\mathbf{x},t)\partial_t\Phi_{\mathbf{p}'''}(\mathbf{x},t) - \partial_t\Phi_{\mathbf{p}''}(\mathbf{x},t)\cdot\Phi_{\mathbf{p}'''}(\mathbf{x},t)] \\
&= i\hbar \int \frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3} \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} f^*(\mathbf{p}')f(\mathbf{p}) \\
&\times \Big[\Phi^\dagger_{\mathbf{p}'}(\mathbf{x},t)\partial_t\Phi_{\mathbf{p}}(\mathbf{x},t) - \partial_t\Phi^\dagger_{\mathbf{p}'}(\mathbf{x},t)\cdot\Phi_{\mathbf{p}}(\mathbf{x},t)\Big] \,.
\end{aligned} \tag{283}$$

We are interested in making this measurement far into the post-acceleration region $\mathcal{M}_+$, where we note that as previously described we may write the mode functions as the plane-wave $\Phi_{\mathbf{p}}(x) = e^{-ip\cdot x/\hbar}$. Hence, in terms of the time-dependence of the mode function we have $\Phi_{\mathbf{p}}(x) \propto e^{-ip_0 t/\hbar}$. Substitution

and a brief rearrangement yield

$$\langle i|\rho(t,\mathbf{x})|i\rangle|_{\mathcal{M}_+} = \frac{1}{2}\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}')f(\mathbf{p})\left(\sqrt{\frac{p_0}{p_0'}}+\sqrt{\frac{p_0'}{p_0}}\right)\Phi^*_{\mathbf{p}'}(\mathbf{x},t)\Phi_{\mathbf{p}}(\mathbf{x},t)\,. \quad (284)$$

Recall that for the sake of simplicity we have defined our coordinate system such that we shall be taking our measurements at time $t = 0$. Using the plane wave mode function, the position expectation value in the direction $x^i(t)$ evaluated at $t = 0$ is

$$\begin{aligned}\langle x^i(0)\rangle &= \frac{1}{2}\int d^3\mathbf{x}\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}')f(\mathbf{p})\left(\sqrt{\frac{p_0}{p_0'}}+\sqrt{\frac{p_0'}{p_0}}\right) x^i e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}/\hbar}\\ &= -\frac{i}{2}\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}')f(\mathbf{p})\left(\sqrt{\frac{p_0}{p_0'}}+\sqrt{\frac{p_0'}{p_0}}\right)\\ &\quad\times\int d^3\mathbf{x}\hbar\partial_{p_i} e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}/\hbar}\,. \end{aligned} \quad (285)$$

Integration of p_i by parts and integration over $\mathbf{x}$ produces

$$\begin{aligned}\langle x^i(0)\rangle &= \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}')\frac{\partial}{\partial p_i}\left[f(\mathbf{p})\left(\sqrt{\frac{p_0}{p_0'}}+\sqrt{\frac{p_0'}{p_0}}\right)\right]\\ &\quad\times(2\pi\hbar)^3\delta(\mathbf{p}-\mathbf{p}')\\ &= \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}(2\pi\hbar)^3\delta(\mathbf{p}'-\mathbf{p})\\ &\quad\times f^*(\mathbf{p}')\left[\frac{\partial}{\partial p_i}f(\mathbf{p})\left(\sqrt{\frac{p_0}{p_0'}}+\sqrt{\frac{p_0'}{p_0}}\right)+\frac{f(\mathbf{p})}{2p_0}\left(\frac{p_i}{\sqrt{p_0p_0'}}-p_i\sqrt{\frac{p_0'}{p_0^3}}\right)\right]\\ &= i\hbar\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p})\frac{\partial}{\partial p_i}f(\mathbf{p})\\ &= \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p})\overleftrightarrow{\partial}_{p_i} f(\mathbf{p})\,. \end{aligned} \quad (286)$$

Due to the fact that the position expectation value is real, the last line takes the real part of the previous one, thus restoring some symmetry to the expression which was lost by the choice of taking the derivative with respect to p_i, as opposed to p_i', of the exponential earlier. We recall that we have chosen to use

the remaining freedom in the choice of coordinate system to arrange the wave packet $f(\mathbf{p})$ such that the position expectation value (286) is equal to 0, viz

$$\langle i|x^i(0)|i\rangle = \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}) \overleftrightarrow{\partial}_{p_i} f(\mathbf{p}) = 0 \quad \forall i = 1,2,3\,. \tag{287}$$

This formula henceforth represents the control against which the position expectation value of the realistic particle whose state has evolved through radiation reaction interactions can be compared. We now duly turn our attention to this evolution.

2. Final interacting state

The 'interacting' particle enters from $\mathcal{M}_-$ with the same initial state as before, namely $|i\rangle$ with the wave packet peaked about some initial momentum $\mathbf{p}_I$. During the accelerations caused by the potential V in the region $\mathcal{M}_I$, the particle, unlike the previous case, is coupled to and interacts with the electromagnetic field. This interaction results in the possible emission of electromagnetic radiation and in radiation reaction effects. Including such interactions to $\mathcal{O}(e^2)$ in $\mathcal{M}_I$, the final out state in $\mathcal{M}_+$ can be either a scalar particle, or a scalar particle and a photon. We designate these two situations the zero-photon and one-photon sectors respectively. In the one-photon sector, the probability amplitude of the emission we, unsurprisingly, call the emission amplitude. The zero-photon sector includes the possibility that the particle does not interact with the electromagnetic field at all but also the one-loop process, the amplitude of which we refer to as the forward scattering amplitude. The Feynman diagrams representing the one loop and emission interactions are presented in Figs. 4.1 and 4.2. The two components of the final state can therefore be written (up to $\mathcal{O}(e^2)$) as

$$|f\rangle_{\rm for} = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3\sqrt{2p_0}}\left[1 + i\mathcal{F}(\mathbf{p})\right] f(\mathbf{p}) A^\dagger(\mathbf{p})|0\rangle \tag{288}$$

$$|f\rangle_{\rm em} = \frac{i}{\hbar}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3\sqrt{2p_0}}\int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) a^\dagger_\mu(\mathbf{k}) f(\mathbf{p}) A^\dagger(\mathbf{P})|0\rangle\,, \tag{289}$$

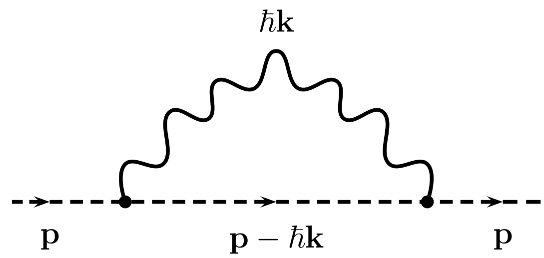

FIGURE 4.1. The one-loop diagram contributing to the forward-scattering amplitude: the dashed and wavy lines represent the scalar and photon propagators, respectively.

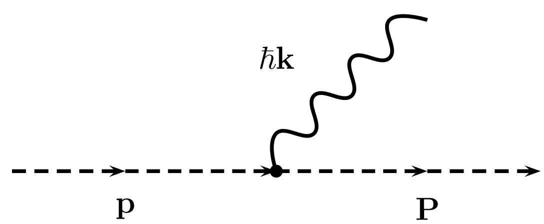

FIGURE 4.2. The one-photon emission diagram contributing to the emission amplitude: the dashed and wavy lines represent the scalar and photon propagators, respectively.

where $\mathcal{F}(\mathbf{p})$ is the forward scattering amplitude and $\mathcal{A}^\mu(\mathbf{p},\mathbf{k})$ the emission amplitude. The reader is urged to note that the momentum of the final scalar particle differs between the two terms due to the energy-momentum carried away by the photon. In the case of a time-dependent potential, we have conservation of momentum and thus $\mathbf{P}=\mathbf{p}-\hbar\mathbf{k}$. If the potential is dependent on one of the spatial directions only, x^3 say, then $\mathbf{P}$ is determined by a combination of energy conservation $\sqrt{\mathbf{p}^2+m^2}=\sqrt{\mathbf{P}+m^2}+\hbar k$ and transverse-momentum conservation $p^i=P^i+\hbar k^i$ $(i=1,2)$.

Comparing these final states with the form of the initial state, where the distribution $f(\mathbf{p})$ was regarded as the one-particle wave function in the momentum representation, let us define

$$F(\mathbf{p}) \equiv [1 + i\mathcal{F}(\mathbf{p})]\, f(\mathbf{p}) \tag{290}$$

$$G^{\mu}(\mathbf{p},\mathbf{k}) \equiv \mathcal{A}^{\mu}(\mathbf{p},\mathbf{k}) f(\mathbf{p})\,. \tag{291}$$

One can then heuristically regard the function $F(\mathbf{p})$ as the one-particle wave function in the zero-photon sector in the $\mathbf{p}$-representation and the function $G^{\mu}(\mathbf{p},\mathbf{k})$ as that in the one-photon sector with a photon with momentum $\hbar\mathbf{k}$ in the $\mathbf{P}$-representation.

The full final state is simply the sum of the above $|f\rangle = |f\rangle_{\mathrm{for}} + |f\rangle_{\mathrm{em}}$. The actual calculation of the forward scattering and emission amplitudes will need to be completed using the mode functions of the field in the region $\mathcal{M}_I$ and will depend upon the circumstances there. We shall return to these calculations later using the semiclassical approximation for the mode functions. In the meantime we can obtain more general expressions for the position expectation value of the final state in terms of these two amplitudes. Later calculations of the amplitudes can then be substituted into the position formulae. We proceed in much the same way that we approached the position of the state $|i\rangle$. As there is no cross term between the two states (288) and (289), the final state density is the sum of the densities of the above states.

2.1. Zero Photon sector. The final state density of the zero photon sector resulting from the forward scattering is given by

$$\begin{aligned} {}_{\mathrm{for}}\langle f|\rho(x)|f\rangle_{\mathrm{for}} = \int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3\sqrt{2p_0'}} \frac{d^3\mathbf{p}}{(2\pi\hbar)^3\sqrt{2p_0}} \langle 0|\, [1 - i\mathcal{F}^*(\mathbf{p}')]\, f^*(\mathbf{p}') A^\dagger(\mathbf{p}') \\ \times \frac{i}{\hbar} : \varphi^\dagger \overleftrightarrow{\partial_t} \varphi : [1 + i\mathcal{F}(\mathbf{p})]\, f(\mathbf{p}) A^\dagger(\mathbf{p}) |0\rangle\,. \end{aligned} \tag{292}$$

Comparison with the calculation pertaining to $|i\rangle$ [1] shows that we have an identical situation after the substitution $f(\mathbf{p}) \to [1 + i\mathcal{F}(\mathbf{p})]\, f(\mathbf{p})$. As a consequence we can write the position expectation value of the state $|f\rangle_{\text{for}}$ at time $t = 0$ as

$$\langle x^i(0)\rangle_{\text{for}} = \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\left([1 - i\mathcal{F}^*(\mathbf{p})]\, f^*(\mathbf{p})\right) \overleftrightarrow{\partial}_{p_i} \left([1 + i\mathcal{F}(\mathbf{p})]\, f(\mathbf{p})\right) . \quad (293)$$

One could therefore regard this state as analogous to the initial 'free' state $|i\rangle$, but with the wave packet distribution $F(\mathbf{p}) \equiv [1 + i\mathcal{F}(\mathbf{p})]\, f(\mathbf{p})$, as opposed to $f(\mathbf{p})$. We note, however, that although $f(\mathbf{p})$ was arranged such that the position expectation value passed through the origin, there is no reason to think the same would be true of $F(\mathbf{p})$. We expand out (293) to order e^2. $\mathcal{F}$ is of order e^2 already and thus we ignore terms at second order in the forward scattering.

$$\begin{aligned}\langle x^i(0)\rangle_{\text{for}} &= \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}) \overleftrightarrow{\partial}_{p_i} f(\mathbf{p}) \left\{1 + i\mathcal{F}(\mathbf{p}) - i\mathcal{F}^*(\mathbf{p})\right\} \\ &+ \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}) f(\mathbf{p}) \partial_{p_i}\left[i\mathcal{F} + i\mathcal{F}^*\right] \\ &= \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\left(f^*(\mathbf{p}) \overleftrightarrow{\partial}_{p_i} f(\mathbf{p})\right)\left[1 - 2\Im\mathcal{F}(\mathbf{p})\right] \\ &- \hbar\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} |f(\mathbf{p})|^2 \partial_{p_i}\Re\mathcal{F} \, . \end{aligned} \quad (294)$$

The reader will undoubtedly have noticed an expression similar to the position expectation value for $|i\rangle$ present in the above. We shall obviously return to this. However, before doing so it will be advantageous to obtain a similar expression for the one photon sector, with which we now proceed.

2.2. One photon sector. In common with the treatment of the forward scattering, we define the position expectation value for the one photon sector

[1] See the first lines of (282).

of the final state as

$$
\begin{aligned}
{}_{\mathrm{em}}\langle f|\rho(x)|f\rangle_{\mathrm{em}} = -\frac{i}{\hbar}\frac{i}{\hbar}\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3\sqrt{2p_0'}}\frac{d^3\mathbf{p}}{(2\pi\hbar)^3\sqrt{2p_0}}\frac{d^3\mathbf{k}'}{2k_0'(2\pi)^3}\frac{d^3\mathbf{k}}{2k_0(2\pi)^3} \\
\langle 0|A(\mathbf{P}')a_\nu(\mathbf{k}')\mathcal{A}^{\nu *}(\mathbf{p}',\mathbf{k}')f^*(\mathbf{p}')\rho(x)\mathcal{A}^\mu(\mathbf{p},\mathbf{k})f(\mathbf{p})a_\mu^\dagger(\mathbf{k})A^\dagger(\mathbf{P})|0\rangle\,,
\end{aligned}
\tag{295}
$$

with $\mathbf{P}'$ defined analogously to $\mathbf{P}$. [2] Using the commutation relations for the electromagnetic field, which we recall are given by

$$
\left[a_\mu(\mathbf{k}), a_\nu^\dagger(\mathbf{k}')\right] = -g_{\mu\nu}(2\pi)^3 2\hbar k\delta^3(\mathbf{k}-\mathbf{k}')\,, \tag{83}
$$

we can write

$$
\langle\rho\rangle_{\mathrm{em}} = -\hbar\int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\langle 0|C_\mu(\mathbf{k})\rho C^{\mu\dagger}(\mathbf{k})|0\rangle\,, \tag{296}
$$

where

$$
C^{\mu\dagger}(\mathbf{k}) \equiv \frac{i}{\hbar}\int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})f(\mathbf{p})A^\dagger(\mathbf{P})\,. \tag{297}
$$

In this form it is easier to see the similarities once again present between the current calculation and that for the initial state. A complication is the difference between $\mathbf{p}$ and $\mathbf{P}$, which we shall now address.

2.2.1. *Space-dependent potential.* In the case of the potential dependent on one of the spatial coordinates[3] - let us choose this coordinate to be x^a - we note that formally we have the transformation $d^3\mathbf{p} = (\partial p_a/\partial P_a)\, d^3\mathbf{P}$. We denote the Jacobian $J_a = (\partial p_a/\partial P_a)$ and stress that this is not a sum as a represents one specific coordinate. We may rewrite $C^{\mu\dagger}$ as

$$
C^{\mu\dagger}(\mathbf{k}) = \frac{i}{\hbar}\int \frac{d^3\mathbf{P}}{\sqrt{2P_0}(2\pi\hbar)^3}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})f(\mathbf{p})A^\dagger(\mathbf{P})\sqrt{\frac{P_0}{p_0}}J_a\,. \tag{298}
$$

[2] i.e. using the same conservation relations albeit with primed variables instead.

[3] The space-dependent case is more complicated than the time dependent case. In this particular calculation it turns out to be more advantageous to perform the complicated version first.

Now, defining

$$g^{\mu}\left(\mathbf{P},\mathbf{k}\right) \equiv \frac{i}{\hbar}\mathcal{A}^{\mu}\left(\mathbf{p},\mathbf{k}\right)f(\mathbf{p})\sqrt{\frac{P_0}{p_0}}J_a\,, \tag{299}$$

we can rewrite (296) as

$$\begin{aligned}\langle\rho\rangle_{\text{em}} = -\hbar\int\frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\\ \langle 0|\int\frac{d^3\mathbf{P}'}{\sqrt{2P_0'}(2\pi\hbar)^3}g^*_{\mu}(\mathbf{P}',\mathbf{k})A(\mathbf{P}')\rho(x)\int\frac{d^3\mathbf{P}}{\sqrt{2P_0}(2\pi\hbar)^3}g^{\mu}(\mathbf{P},\mathbf{k})A^{\dagger}(\mathbf{P})|0\rangle\,.\end{aligned} \tag{300}$$

Comparison with (282) is now clear. The calculation for $|i\rangle$ may then be followed for the position expectation value of the state $|f\rangle_{\text{em}}$ to give

$$\langle x^i\rangle_{\text{em}} = -\frac{i\hbar^2}{2}\int\frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\frac{d^3\mathbf{P}}{(2\pi\hbar)^3}\left(g^*_{\mu}(\mathbf{P},\mathbf{k})\overleftrightarrow{\partial}_{P_i}g^{\mu}(\mathbf{P},\mathbf{k})\right)\,. \tag{301}$$

Returning to our original notation and using the symmetry of the derivative operator $\overleftrightarrow{\partial}_{P_i}$, we find

$$\begin{aligned}\langle x^i\rangle_{\text{em}} = -\frac{i}{2}\int\frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\frac{d^3\mathbf{P}}{(2\pi\hbar)^3}\\ \times\left(\mathcal{A}^*_{\mu}(\mathbf{p},\mathbf{k})f^*(\mathbf{p})\overleftrightarrow{\partial}_{P_i}\mathcal{A}^{\mu}(\mathbf{p},\mathbf{k})f(\mathbf{p})\right)\frac{P_0}{p_0}J_a^2\,.\end{aligned} \tag{302}$$

Converting the integration variable back from $\mathbf{P}$ to $\mathbf{p}$ and changing the P_i-derivative to a p_i-derivative we produce

$$\begin{aligned}\langle x^i(0)\rangle_{\text{em}} = -\frac{i}{2}\int\frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\\ \times\left(\mathcal{A}^*_{\mu}(\mathbf{p},\mathbf{k})f^*(\mathbf{p})\overleftrightarrow{\partial}_{p_i}\mathcal{A}^{\mu}(\mathbf{p},\mathbf{k})f(\mathbf{p})\right)\frac{P_0}{p_0}J_a\frac{\partial p_i}{\partial P_i}\,.\end{aligned} \tag{303}$$

Separating out the emission amplitude and wave packet distribution terms, the position expectation value for the emission state is given by two terms:

$$\begin{aligned}\langle x^i(0)\rangle_{\text{em}} = -\frac{i}{2}\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}f^*(\mathbf{p})\overleftrightarrow{\partial}_{p_i}f(\mathbf{p})\\ \times\int\frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k})\mathcal{A}_{\mu}(\mathbf{p},\mathbf{k})\frac{P_0}{p_0}J_a\frac{\partial p_i}{\partial P_i}\\ -\frac{i}{2}\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\int\frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k})\overleftrightarrow{\partial}_{p_i}\mathcal{A}_{\mu}(\mathbf{p},\mathbf{k})\,.\end{aligned} \tag{304}$$

We have dropped the factor $(P_0/p_0)\, J_a\, (\partial p_i/\partial P_i)$ from the second term for the following reasons: The emission amplitude $\mathcal{A}^\mu$ and its p_i-derivative are both[4] of order $\hbar^0$, and thus so is the second term in the above expression as it is now written. To order $\hbar^0$, $P_0 = p_0$ and $\partial p_i/\partial P_i = 1$ for all $i = 1, 2, 3$ (including a), hence $J_a = 1$ to lowest order. Consequently we may replace these terms with unity at this order. However, one needs to keep these factors in the first term of (304), which is of order $\hbar^{-1}$.[5]

2.2.2. *Time-dependent potential.* Returning to the expression for the charge density/probability density expectation value (296), we proceed with the simpler case of the time-dependent potential. The definition of $C^{\mu\dagger}$ from (297) still holds, which we repeat as

$$C^{\mu\dagger}(\mathbf{k}) \equiv \frac{i}{\hbar} \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} \mathcal{A}^\mu(\mathbf{p}, \mathbf{k}) f(\mathbf{p}) A^\dagger(\mathbf{P})\,. \tag{297}$$

The time-dependent potential is simpler because the conservation of momentum means that we can write $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$. Consequently we have $d^3\mathbf{P} = d^3\mathbf{p}$. In connection with the previous workings we can write

$$C^{\mu\dagger}(\mathbf{k}) \equiv \int \frac{d^3\mathbf{P}}{\sqrt{2P_0}(2\pi\hbar)^3} g_t^\mu(\mathbf{P}, \mathbf{k}) A^\dagger(\mathbf{P})\,, \tag{305}$$

where this time the momentum distribution is given by

$$g_t^\mu(\mathbf{P}, \mathbf{k}) \equiv \frac{i}{\hbar} \mathcal{A}^\mu(\mathbf{p}, \mathbf{k}) \sqrt{\frac{P_0}{p_0}}\,. \tag{306}$$

[4]This is demonstrated in section 3 in the calculation of the emission amplitude using the semiclassical approximation (given in (344) or (359)).

[5]This can be seen from the previous footnote, on the emission amplitude, and by comparison with the expression in (287).

The difference with respect to the previous case is the absence of the $\partial p_a/\partial P_a$ factor. With reference to this we see that we have

$$
\begin{aligned}
\langle x^i\rangle_{\text{em}} &= -\frac{i\hbar^2}{2}\int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\frac{d^3\mathbf{P}}{(2\pi\hbar)^3}\left(g^*_{t\mu}(\mathbf{P},\mathbf{k})\overleftrightarrow{\partial}_{P_i}\, g^\mu_t(\mathbf{P},\mathbf{k})\right)\\
&= -\frac{i}{2}\int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\left(\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})f^*(\mathbf{p})\overleftrightarrow{\partial}_{p_i}\,\mathcal{A}^\mu(\mathbf{p},\mathbf{k})f(\mathbf{p})\right)\frac{P_0}{p_0}\frac{\partial p_i}{\partial P_i}\\
&= -\frac{i}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}f^*(\mathbf{p})\overleftrightarrow{\partial}_{p_i}\, f(\mathbf{p})\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu*}(\mathbf{p},\mathbf{k})\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\frac{P_0}{p_0}\frac{\partial p_i}{\partial P_i}\\
&\quad -\frac{i}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu*}(\mathbf{p},\mathbf{k})\overleftrightarrow{\partial}_{p_i}\,\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\,.
\end{aligned}
\tag{307}
$$

The previous arguments pertaining to the order of the two terms are fully applicable here too. We see that the only difference between the two is the removal of the $\partial p_a/\partial P_a$ type factor when moving from $V(x^a)$ to $V(t)$.

Considering $\langle x^i\rangle_{\text{em}}$ in (304) and (307), the ever observant reader will once again note the similarity between the first term and the position expectation value for $|i\rangle$. When this was noted for the forward scattering, we delayed consideration of the factor until after the corresponding emission calculation had been performed. We now return as promised to consider these two terms.

2.3. Normalisation and unitarity. The final state $|f\rangle$ for the interacting particle is the sum of the components $|f\rangle_{\text{em}}$ and $|f\rangle_{\text{for}}$. Whilst we have already observed that there is no cross term, there is however a connection to be made using the normalisation condition for $|f\rangle$. Recalling the definition and normalisation of $|i\rangle$ and using the unitarity of time evolution, we find

$$\langle f|f\rangle = 1 \tag{308}$$

In other words, the normalisation of the two components must also add to unity. For the forward scattering zero-photon sector, we have

$$
\begin{aligned}
{}_{\text{for}}\langle f|f\rangle_{\text{for}} &= \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3} \\
&\qquad \langle 0|A(\mathbf{p}')f(\mathbf{p}')\left[1-i\mathcal{F}^*(\mathbf{p}')\right]\left[1+i\mathcal{F}(\mathbf{p})\right]f(\mathbf{p})A^\dagger|0\rangle \\
&= \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\left[1-2\Im\mathcal{F}(\mathbf{p})\right]+\mathcal{O}(e^4)\,.
\end{aligned} \tag{309}
$$

The last term is added as a reminder that terms are taken to first order in e^2. For the one-photon sector, the inner product produces the emission probability $\mathcal{P}_{\text{em}}$ viz

$$
\begin{aligned}
{}_{\text{em}}\langle f|f\rangle_{\text{em}} &= \mathcal{P}_{\text{em}} \\
&= -\frac{i}{\hbar}\frac{i}{\hbar}\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3\sqrt{2p_0'}}\frac{d^3\mathbf{p}}{(2\pi\hbar)^3\sqrt{2p_0}}\frac{d^3\mathbf{k}'}{2k_0'(2\pi)^3}\frac{d^3\mathbf{k}}{2k_0(2\pi)^3} \\
&\qquad \langle 0|A(\mathbf{P}')a_\nu(\mathbf{k}')\mathcal{A}^{\nu *}(\mathbf{p}',\mathbf{k}')f^*(\mathbf{p}')\rho(x)\mathcal{A}^\mu(\mathbf{p},\mathbf{k})f(\mathbf{p})a^\dagger_\mu(\mathbf{k})A^\dagger(\mathbf{P})|0\rangle \\
&= -\hbar\int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\langle 0|C_\mu(\mathbf{k})C^{\mu\dagger}(\mathbf{k})|0\rangle\,,
\end{aligned} \tag{310}
$$

using the definition of $C^{\mu\dagger}(\mathbf{k})$ in (297). This equation then compares with (296) and we shall require similar manipulations of $\mathbf{p}$ and $\mathbf{P}$. We have

$$
\begin{aligned}
\mathcal{P}_{\text{em}} &= -\frac{1}{\hbar}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3} \\
&\qquad f^*(\mathbf{p}')\mathcal{A}^*_\mu(\mathbf{p}',\mathbf{k})f(\mathbf{p})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\langle 0|A(\mathbf{P}')A^\dagger(\mathbf{P})|0\rangle\,.
\end{aligned} \tag{311}
$$

In order to perform the $\mathbf{p}'$ integration in the case of the potential dependent on spatial direction x^a, we note that from the commutation relations we have

$$
\begin{aligned}
\langle 0|A(\mathbf{P}')A^\dagger(\mathbf{P})|0\rangle &= 2P_0(2\pi\hbar)^3\delta^3(\mathbf{P}-\mathbf{P}') \\
&= 2p_0(2\pi\hbar)^3\delta^3(\mathbf{p}-\mathbf{p}')\frac{P_0}{p_0}\frac{\partial p_a}{\partial P_a}\,.
\end{aligned} \tag{312}
$$

Thus the emission probability can be written

$$
\mathcal{P}_{\text{em}} = -\frac{1}{\hbar}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\frac{P_0}{p_0}\frac{\partial p_a}{\partial P_a}\,. \tag{313}
$$

Given that (313) and (309) must sum to unity for all $f(\mathbf{p})$, and recalling the normalisation[6] of $f(\mathbf{p})$, we must have

$$2\Im\mathcal{F}(\mathbf{p}) = -\frac{1}{\hbar}\int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\frac{P_0}{p_0}\frac{\partial p_a}{\partial P_a}\,. \tag{314}$$

Via a similar argument presented in the derivation of $\langle x^i\rangle_{\mathrm{em}}$ in (304) and (307), we note that for the t-dependent potential, the appropriate expressions are

$$\mathcal{P}_{\mathrm{em}} = -\frac{1}{\hbar}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\frac{P_0}{p_0}\,, \tag{315}$$

and

$$2\Im\mathcal{F}(\mathbf{p}) = -\frac{1}{\hbar}\int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\frac{P_0}{p_0}\,. \tag{316}$$

2.4. Position expectation value. Combining the position expectation values from the two components of the final state as given in (294) and (304) we have (using the subscript f to denote the full final state),

$$\begin{aligned}\langle x^i(0)\rangle_{\mathrm{f}} = {} & \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\left(f^*(\mathbf{p})\overleftrightarrow{\partial}_{p_i} f(\mathbf{p})\right)[1-2\Im\mathcal{F}(\mathbf{p})]\\ & -\hbar\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\partial_{p_i}\Re\mathcal{F}\\ & -\frac{i}{2}\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}f^*(\mathbf{p})\overleftrightarrow{\partial}_{p_i} f(\mathbf{p})\int\frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k})\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\frac{P_0}{p_0}\frac{\partial p_a}{\partial P_a}\frac{\partial p_i}{\partial P_i}\\ & -\frac{i}{2}\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\int\frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k})\overleftrightarrow{\partial}_{p_i}\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\,,\end{aligned} \tag{317}$$

for which we are using the space-dependent potential expressions. The relation in (314) can be used to eliminate the imaginary part of the forward scattering.

[6]The expression for the normalisation $\langle i|i\rangle = 1$ in (75).

The result can be written in three terms:

$$\begin{aligned}
&\langle x^i(0)\rangle_{\mathrm{f}} \\
&= \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\left(f^*(\mathbf{p})\overleftrightarrow{\partial}_{p_i} f(\mathbf{p})\right) \\
&\quad + \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\left[-\hbar\partial_{p_i}\Re\mathcal{F} - \frac{i}{2}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k})\overleftrightarrow{\partial}_{p_i}\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\right] \\
&\quad - \frac{i}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\left(f^*(\mathbf{p})\overleftrightarrow{\partial}_{p_i} f(\mathbf{p})\right) \\
&\qquad \times \int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k})\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\frac{P_0}{p_0}\frac{\partial p_a}{\partial P_a}\left(\frac{\partial p_i}{\partial P_i} - 1\right). \qquad (318)
\end{aligned}$$

Now our task is to interpret these terms. The first term can be recognized as the position expectation value of the non-interacting state $|i\rangle$, which is written $\langle x^i(0)\rangle_{\mathrm{i}}$. As this is our control particle, from the definition of the position shift as the change in position due to radiation reaction effects, we conclude that the position shift is the sum of the second and third terms. The reader will recall that the function $f(\mathbf{p})$ was defined to be sharply peaked about the final momentum. Calling this final momentum $\mathbf{p}$ for simplicity[7], we find that in the $\hbar \to 0$ limit, these two terms are given by

$$\delta x^i_{\mathrm{Q1}} = -\hbar\partial_{p_i}\Re\mathcal{F} - \frac{i}{2}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k})\overleftrightarrow{\partial}_{p_i}\mathcal{A}_\mu(\mathbf{p},\mathbf{k}), \qquad (319)$$

$$\delta x^i_{\mathrm{Q2}} = -\frac{\langle x^i(0)\rangle_{\mathrm{i}}}{\hbar}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k})\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\left(\frac{\partial p_i}{\partial P_i} - 1\right). \qquad (320)$$

For δx^i_{Q2} we have again the fact that $\partial p_i/\partial P_i - 1$ is of order $\hbar$ and consequently in this $\hbar \to 0$ limit dropped the factor $(P_0/p_0)(\partial p_a/\partial P_a)$. Dropping this factor also means that we may continue with the results applying to both time and space dependent potential cases. The quantum position shift δx^i_{Q} due to radiation reaction of the scalar field can thus be written

$$\delta x^i_{\mathrm{Q}} = \delta x^i_{\mathrm{Q1}} + \delta x^i_{\mathrm{Q2}}. \qquad (321)$$

[7]This is of course equivalent to the momentum being peaked about $\tilde{\mathbf{p}}$ say, followed by a change of variables $\tilde{\mathbf{p}} \to \mathbf{p}$.

Now, in the choice of coordinate system that we have chosen we have set $\langle x^i(0)\rangle_{\mathrm{i}} = 0$, and thus $\delta x^i_{\mathrm{Q2}} = 0$ at this order. We have nevertheless kept this contribution up to now in order to obtain a formula for the position expectation value without making this assumption which would be given, in terms of the above definitions, to order $\hbar$ by

$$\langle x^i(0)\rangle_{\mathrm{f}} = \langle x^i(0)\rangle_{\mathrm{i}} + \delta x^i_{\mathrm{Q1}} + \delta x^i_{\mathrm{Q2}} \,. \tag{322}$$

Indeed, we shall later show that δx^i_{Q2} gives the correct correction to the position expectation value if we use our freedom of coordinate system to choose $\langle x^i(0)\rangle_{\mathrm{i}} \neq 0$.

Returning for now to our standard choice of coordinates, we find that the quantum position shift is given by $\delta x^i_{\mathrm{Q}} = \delta x^i_{\mathrm{Q1}}$ or

$$\delta x^i_{\mathrm{Q}} = -\hbar \partial_{p_i} \Re \mathcal{F} - \frac{i}{2} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial}_{p_i} \mathcal{A}_\mu(\mathbf{p},\mathbf{k}) \,. \tag{323}$$

There are two contributions to this shift, coming naturally from the forward scattering and emission contributions to the final state. In order to analyse the quantum position shift and compare the result with the classical position shift δx^i_{C} given in (279), we must calculate these contributions. For this purpose firstly define

$$\delta x^i_{\mathrm{for}} = -\hbar \partial_{p_i} \Re \mathcal{F}(\mathbf{p}) \,, \tag{324}$$

$$\delta x^i_{\mathrm{em}} = -\frac{i}{2} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial}_{p_i} \mathcal{A}_\mu(\mathbf{p},\mathbf{k}) \,, \tag{325}$$

as the quantum position shift due to forward scattering and emission respectively.

3. Emission Amplitude

We now come to the task of calculating the emission amplitude. Recall that the emission amplitude was originally defined by its presence in the one-photon sector of the final state which we repeat here (with the original equation

numbering)

$$|f\rangle_{\mathrm{em}} = \frac{i}{\hbar}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3\sqrt{2p_0}}\int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})a^\dagger_\mu(\mathbf{k})f(\mathbf{p})A^\dagger(\mathbf{P})|0\rangle\,. \qquad (289)$$

Considering the form of the initial state, the above represents the following state evolution

$$A^\dagger(\mathbf{p})|0\rangle \to \cdots + \frac{i}{\hbar}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})a^\dagger_\mu(\mathbf{k})A^\dagger(\mathbf{P})|0\rangle\,. \qquad (326)$$

As it stands this is merely a definition for $\mathcal{A}^\mu$. The full first order evolution of this state in time-dependent perturbation theory is

$$A^\dagger(\mathbf{p})|0\rangle \to \cdots - \frac{i}{\hbar}\int d^4x\mathcal{H}_I(x)A^\dagger(\mathbf{p})|0\rangle\,, \qquad (327)$$

where $\mathcal{H}_I(x)$ is the interaction Hamiltonian density. Comparing these two evolution expressions, we can write the emission amplitude in terms of the interaction Hamiltonian density as

$$\mathcal{A}_\mu(\mathbf{p},\mathbf{k}) = \frac{1}{\hbar}\int \frac{d^3\mathbf{p}'}{2p'_0(2\pi\hbar)^3}\int d^4x\langle 0|a_\mu(\mathbf{k})A(\mathbf{p}')\mathcal{H}_I(x)A^\dagger(\mathbf{p})|0\rangle\,. \qquad (328)$$

The Hamiltonian density for scalar electrodynamics[8] can be obtained from the Lagrangian density by the standard method and was given in (85) and we repeat it here to aid the reader:[9]

$$\mathcal{H}_I(x) = \frac{ie}{\hbar}A_\mu{:}\big[\varphi^\dagger D^\mu\varphi - (D^\mu\varphi)^\dagger\varphi\big]{:}+\frac{e^2}{\hbar^2}\sum_{i=1}^{3}A_iA_i{:}\varphi^\dagger\varphi{:}\,. \qquad (85)$$

As was noted when this expression was originally introduced, it is useful to observe the $i = 1, 2, 3$ sum in the second term i.e. the absence of the A_0A_0-type term. The first term is the contraction between the electromagnetic field

[8]A note should be made at this point that we are using the normal ordering from the free-field. The appropriate subtraction to be made is technically the subtraction of the vacuum in the $V \to 0$ limit of the potential. However, it can be shown that to at least order $\hbar^2$ the difference between the methods is zero [**9**] and consequently we are justified in using the more familiar normal ordering here.

[9]Normal ordering on the A_iA_i type term would add an infinite constant term altering our definition of the mass counter-term and would not affect the final result. The full treatment of both the electromagnetic fields and scalar fields is technically that noted in the previous footnote and gives the same results as this treatment [**9**].

and the current of the scalar field $J^\mu(x)$. It is this term with which we are presently interested as it is the current-EM field coupling that produces the photon emission process with which we may match the emission amplitude expression. Thus, we have

$$\begin{aligned}&\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\\&= \frac{ie}{\hbar^2}\int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3}\int d^4x \langle 0|a_\mu(\mathbf{k})A(\mathbf{p}')\left\{A_\nu\!:\!\left[\varphi^\dagger D^\nu\varphi-(D^\nu\varphi)^\dagger\varphi\right]\!:\!\right\}A^\dagger(\mathbf{p})|0\rangle\,.\end{aligned} \tag{329}$$

By using the expansion of the fields A_μ and φ, and the commutation relations for the annihilation and creation operators, we readily find

$$\begin{aligned}&\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\\&= -ie\hbar\int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3}\int d^4x\, e^{ik\cdot x}\left\{\Phi^\dagger_{\mathbf{p}'}(x)D_\mu\Phi_\mathbf{p}(x)-\left[D_\mu\Phi_{\mathbf{p}'}(x)\right]^\dagger\Phi_\mathbf{p}(x)\right\}\,.\end{aligned} \tag{330}$$

We may now proceed to substitute the appropriate expression for the mode function into the above and calculate $\mathcal{A}^\mu$.

3.1. Time dependent potential. We begin by looking at the case of a time-dependent potential $V^i(t)$ $(i=1,2,3)$ with $V^0(t)=0$. [10] The system is translationally invariant in the spatial directions and hence we can let

$$\Phi_\mathbf{p}(\mathbf{x},t)=\phi_\mathbf{p}(t)\exp\left(i\mathbf{p}\cdot\mathbf{x}/\hbar\right)\,. \tag{331}$$

The amplitude in a spatial direction for the t-dependent potential is then

$$\begin{aligned}\mathcal{A}^i(\mathbf{p},\mathbf{k}) &= -e\int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3}\int d^4x\,\phi^*_{\mathbf{p}'}(t)\phi_\mathbf{p}(t)\left[p^i+p'^i-2V^i(t)\right]e^{i[(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}/\hbar]}\\&\qquad\times e^{i(kt-\mathbf{k}\cdot\mathbf{x})}\\&= -e\int dt\, e^{ikt}\phi^*_\mathbf{p}(t)\phi_\mathbf{p}(t)\frac{p^i-V^i(t)}{p_0}\,,\end{aligned} \tag{332}$$

[10] If $V^0(t)\neq 0$, one is free to gauge away this component. Our choice here is for consistency with $\partial_\mu A^\mu=0$.

with $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$, where we have let $P^i + p^i - 2V^i(t) = 2[p^i - V(t)]$ as the difference $p^i - P^i$ is of order $\hbar$. [11] However, it would be incorrect to equate $\phi_{\mathbf{P}}(t)$ with $\phi_{\mathbf{p}}(t)$, because these functions oscillate with periods of order $\hbar^{-1}$, as will be seen shortly. For the time component we have $D_t = \partial_t$ and simply obtain

$$\mathcal{A}^0(\mathbf{p}, \mathbf{k}) = -\frac{ie\hbar}{2p_0} \int dt\, \left[\phi^*_{\mathbf{P}} \partial_t \phi_{\mathbf{p}} - (\partial_t \phi^*_{\mathbf{P}})\, \phi_{\mathbf{p}}\right] e^{ikt}\,. \tag{333}$$

To proceed further, we require the remaining undetermined factor $\phi_{\mathbf{p}}(t)$ of the mode function to be approximated for the field in the region $\mathcal{M}_I$ in a form suitable for taking the $\hbar \to 0$ limit. This is of course precisely what we have in the semiclassical approximation in (142). We repeat this result here to aid the reader:

$$\phi_{\mathbf{p}}(t) = \sqrt{\frac{p_0}{E_p(t)}} \exp\left[-\frac{i}{\hbar}\int_0^t E_p(\zeta) d\zeta\right]\,, \tag{334}$$

where

$$E_p(t) \equiv \sqrt{|\mathbf{p} - \mathbf{V}(t)|^2 + m^2}\,. \tag{335}$$

We note that the local momentum and energy of the point particle corresponding to the wave packet considered here are

$$m\frac{d\mathbf{x}}{d\tau} = \mathbf{p} - \mathbf{V}(t)\,, \tag{336}$$

$$m\frac{dt}{d\tau} = E_p(t)\,. \tag{337}$$

Now, the product of two wave functions in the emission amplitude (332) can be written

$$\phi^*_{\mathbf{P}}(t)\phi_{\mathbf{p}}(t) = \frac{p_0}{E_p(t)} \exp\left\{-\frac{i}{\hbar}\int_0^t \left[E_p(\zeta) - E_P(\zeta)\right] d\zeta\right\}\,, \tag{338}$$

[11] Later, when we consider the forward-scattering, the semiclassical approximation for the emission probability is justified (475). This in turn shows the validity of the physically reasonable assumption that a typical photon emitted has energy of order $\hbar$.

where we have replaced P_0 and $E_P(t)$ in the pre-factor by p_0 and $E_p(t)$, respectively, due to the $\hbar \to 0$ limit. [12] The integrand in the exponent can be evaluated to lowest order in $\hbar$ by using (336) and (337) as

$$\begin{aligned} E_p - E_P &= \frac{\partial E_p}{\partial p^i}(P^i - p^i) \\ &= \frac{dx^i}{dt}\hbar k^i\,, \end{aligned} \tag{339}$$

where the repeated indices i are summed over. By substituting this approximation in (338) we find

$$\begin{aligned} \phi^*_{\mathbf{P}}(t)\phi_{\mathbf{p}}(t) &= \frac{p_0}{E_p}\exp\left(-i\int_0^t dt\,\frac{dx^i}{dt}k^i\right) \\ &= \frac{p_0}{E_p}\exp\left(-i\mathbf{k}\cdot\mathbf{x}\right)\,, \end{aligned} \tag{340}$$

where we have used the fact that the particle passes through the spacetime origin. By substituting this formula in (332) and noting (336) and (337) we obtain

$$\begin{aligned} \mathcal{A}^i(\mathbf{p},\mathbf{k}) &= -e\int dt\, e^{ik\cdot x}\frac{dx^i}{dt} \\ &= -e\int d\xi\,\frac{dx^i}{d\xi}e^{ik\xi}\,, \end{aligned} \tag{341}$$

where we have defined $\xi \equiv t - \mathbf{n}\cdot\mathbf{x}$ with $\mathbf{n} \equiv \mathbf{k}/k$. We emphasize that $\mathbf{x}$ and ξ here are functions of t evaluated on the world line of the corresponding classical particle passing through the spacetime origin.

Let us now consider the time component $\mathcal{A}^0(\mathbf{p},\mathbf{k})$ of the emission amplitude given by (333). Note that from the semiclassical expression (334) for $\phi_{\mathbf{p}}(t)$ we have, to lowest order in $\hbar$,

$$\partial_t\phi_{\mathbf{p}}(t) = -\frac{i}{\hbar}E_p(t)\phi_{\mathbf{p}}(t)\,. \tag{342}$$

[12]This is not contrary to our previous point regarding the order of the product as the oscillation period, which is of $\mathcal{O}(\hbar^{-1})$, is contained in the exponential. The pre-factor replacements may thus be made analogously with those multiplicative factors in (332).

By substituting this formula in (333) we obtain

$$\begin{aligned}\mathcal{A}^0(\mathbf{p},\mathbf{k}) &= -\frac{e}{2p_0}\int dt\ [E_p(t)+E_P(t)]\,\phi^*_{\mathbf{P}}(t)\phi_{\mathbf{p}}(t)e^{ikt} \\ &= -e\int dt\, e^{-i\mathbf{k}\cdot\mathbf{x}}e^{ikt} \\ &= -e\int d\xi\,\frac{dt}{d\xi}e^{ik\xi}\,, \end{aligned} \tag{343}$$

where we have let $E_P(t) = E_p(t)$ and used (340). By combining this formula and (341) we obtain the following concise expression for the $\hbar \to 0$ limit of the emission amplitude:

$$\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) = -e\int d\xi\,\frac{dx^\mu}{d\xi}e^{ik\xi}\,. \tag{344}$$

3.2. Space-dependent potential. Let us now consider the case where the potential is dependent on one of the spatial coordinates, z say, although the following will apply to x and y equally by symmetry. The following calculations are very similar to the previous t-dependent potential case, albeit with subtle differences in the workings. We once again start from the equation (330)

$$\begin{aligned}&\mathcal{A}_\mu(\mathbf{p},\mathbf{k}) \\ &= -ie\hbar\int\frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3}\int d^4x\, e^{ik\cdot x}\left[\Phi^\dagger_{\mathbf{p}'}(x)D_\mu\Phi_{\mathbf{p}}(x) - (D_\mu\Phi_{\mathbf{p}'}(x))^\dagger\,\Phi_{\mathbf{p}}(x)\right]\,.\end{aligned} \tag{330}$$

The potential is $V^\mu(z)$, with $V^3(z) = 0$. The translational invariance in the t, x and y directions means that the mode function can be decomposed by

$$\Phi_{\mathbf{p}} = \phi_{\mathbf{p}}(z)e^{-\frac{i}{\hbar}(p_0t-p_xx-p_yy)}\,. \tag{345}$$

Here let us use the notation $\perp$ to represent the x, y directions. The amplitude for the x, y components is

$$\begin{aligned}
\mathcal{A}_\perp &= e \int \frac{d^3p}{2p'_0(2\pi\hbar)^3} \int d^4x \, \phi^*_{\mathbf{p}'}(z)\phi_{\mathbf{p}}(z) \left[p_\perp + p'_\perp - 2V_\perp(z)\right] \\
&\qquad \times e^{i((\mathbf{p}_\perp - \mathbf{p'}_\perp)\cdot \mathbf{x}_\perp)/\hbar} e^{-i(p_0 - p'_0)t} e^{i(k_0 t - \mathbf{k}\cdot\mathbf{x})} \\
&= e \int \frac{d^3p}{2p'_0(2\pi\hbar)^3} \int dz \, \phi^*_{\mathbf{p}'}(z)\phi_{\mathbf{p}}(z) \left[p_\perp + p'_\perp - 2V_\perp(z)\right] e^{-ik_z z} (2\pi\hbar)^3 \\
&\qquad \times \delta(p'_0 + \hbar k_0 - p_0)\delta^2(p'_\perp + \hbar k_\perp - p_\perp) \\
&= e \int dz \, e^{-ik_z z} \phi^*_{\mathbf{P}}(z)\phi_{\mathbf{p}}(z) \frac{\left[p_\perp - V_\perp(z)\right]}{p_z} \, ,
\end{aligned} \tag{346}$$

where in the last two lines we firstly integrated over x, y, t then used the fact that $dp_z/p_0 = dp_0/p_z$ before integrating out the delta functions. For the last line, P is defined via the conservation of transverse momentum and energy represented by the delta functions, i.e. $\mathbf{P}_\perp = \mathbf{p}_\perp - \hbar\mathbf{k}_\perp$ and $\sqrt{\mathbf{P}^2 + m^2} = \sqrt{\mathbf{p}^2 + m^2} - \hbar k_0$. Consequently, following analogous reasoning set out in the t-dependent potential evaluation, we take $P_\perp + p_\perp - 2V_\perp(z) = 2\left[p_\perp - V_\perp(z)\right]$ as the difference is of order $\hbar$. Once again, $\phi^*_{\mathbf{P}}(z)\phi_{\mathbf{p}}(z)$ must be treated carefully. For the t component the only difference to the above is the covariant derivative D_t gives an extra minus sign. This gives

$$\mathcal{A}_t = -e \int dz \, e^{-ik_z z} \phi^*_{\mathbf{p}'}(z)\phi_{\mathbf{p}}(z) \frac{\left[p_0 - V_t(z)\right]}{p_z} \, . \tag{347}$$

For the z component we have $D_z = \partial_z$ thus simply obtain

$$\begin{aligned}
\mathcal{A}_z &= -ie\hbar \int \frac{d^3p}{2p'_0(2\pi\hbar)^3} \int d^4x \, \left[\phi^*_{\mathbf{p}'}\partial_z\phi_{\mathbf{p}} - \left(\partial_z\phi^*_{\mathbf{p}'}\right)\phi_{\mathbf{p}}\right] \\
&\qquad \times e^{-\frac{i}{\hbar}\left((p_0 - p'_0)t - (p_x - p'_x)x - (p_y - p'_y)y\right)} e^{i(k_0 t - \mathbf{k}\cdot\mathbf{x})} \\
&= -\frac{ie\hbar}{2p_z} \int dz \, \left[\phi^*_{\mathbf{p}'}\partial_z\phi_{\mathbf{p}} - \left(\partial_z\phi^*_{\mathbf{p}'}\right)\phi_{\mathbf{p}}\right] e^{-ik_z z} \, ,
\end{aligned} \tag{348}$$

where we have again changed the integration from p_z to p_0. This expression is similar to the case $\mathcal{A}_0$ for $V(t)$ in (333).

We now need the semiclassical expression for the remainder of the mode function substitution. This was found in (153) and as before, we reproduce it

here to aid the reading of this calculation.

$$\phi_{\mathbf{p}}(t) = \sqrt{\frac{p_z}{\kappa_{\mathbf{p}}(z)}} \exp\left[\frac{i}{\hbar}\int_0^z \kappa_{\mathbf{p}}(\zeta) d\zeta\right] , \tag{349}$$

where

$$\kappa_p(z) = \sqrt{(p_0 - V_t(t))^2 - (p_x - V_x(t))^2 - (p_y - V_y(t))^2 - m^2} . \tag{350}$$

This time we note that $(p_\perp - V_\perp(z))$,$(p_0 - V_t(z))$ are the i,t components ($\perp = x, y$) of the momentum of the corresponding classical particle and κ_p is the z component, i.e. $dx^\perp/d\tau$,$dt/d\tau$ and $dz/d\tau$ respectively. The product expression is now written

$$\phi^*_{\mathbf{P}}(z)\phi_{\mathbf{p}}(z) = \sqrt{\frac{P_z p_z}{\kappa_{\mathbf{P}}(z)\kappa_{\mathbf{p}}(z)}} \exp\left[\frac{i}{\hbar}\int_0^z \left(\kappa_{\mathbf{p}}(\zeta) - \kappa_{\mathbf{P}}(\zeta)\right) d\zeta\right] . \tag{351}$$

Again, to lowest order in $\hbar$ we can change P_z to p_z and $\kappa_{\mathbf{P}}(z)$ to $\kappa_{\mathbf{p}}(z)$ as the difference is of order $\hbar$. The difference in the κ terms to lowest order in $\hbar$ is

$$\begin{aligned} \kappa_{\mathbf{p}} - \kappa_{\mathbf{P}} &= \frac{\partial \kappa_{\mathbf{p}}}{\partial p_0}(P_0 - p_0) + \frac{\partial \kappa_{\mathbf{p}}}{\partial \mathbf{p}_\perp} \cdot (\mathbf{P}_\perp - \mathbf{p}_\perp) \\ &= \frac{dt}{dz}\hbar k_0 - \frac{d\mathbf{x}_\perp}{dz} \cdot \hbar \mathbf{k}_\perp . \end{aligned} \tag{352}$$

Thus the product of ϕ's can be written

$$\begin{aligned} \phi^*_{\mathbf{p}'}(z)\phi_{\mathbf{p}}(z) &= \frac{p_z}{\kappa_{\mathbf{p}}} \exp\left[\frac{i}{\hbar}\int_0^z dz\, \frac{dt}{dz}\hbar k_0 - \frac{dx_\perp}{dz}\hbar k_\perp\right] \\ &= \frac{p_z}{\kappa_{\mathbf{p}}} \exp\left(i(k_0 t - k_\perp \cdot x_\perp)\right) . \end{aligned} \tag{353}$$

This gives the i component of the emission amplitude to be

$$\begin{aligned} \mathcal{A}_i &= e \int dz\, e^{-ik_z z} \frac{p_z}{\kappa_{\mathbf{p}}} \exp\left(i(k_0 t - k_\perp \cdot x_\perp)\right) \frac{[p_i - V_i(z)]}{p_z} \\ &= e \int d\xi\, \frac{dx_i}{d\xi} e^{ik\xi} , \end{aligned} \tag{354}$$

using $\xi = t - \mathbf{n}\cdot\mathbf{x}$, $\mathbf{n} = \mathbf{k}/k$ as defined previously. Similarly, the t component gives

$$\mathcal{A}_t = -e \int d\xi\, \frac{dx_i}{d\xi} e^{ik\xi} . \tag{355}$$

Consider now the z component of the amplitude given previously as

$$\mathcal{A}_z = -\frac{ie\hbar}{2p_z}\int dz\ \left[\phi^*_{\mathbf{p}'}\partial_z\phi_{\mathbf{p}} - \left(\partial_z\phi^*_{\mathbf{p}'}\right)\phi_{\mathbf{p}}\right] e^{-ik_z z}\,. \tag{356}$$

Note here that from equation for $\phi_{\mathbf{p}}$ we have, for lowest order in $\hbar$

$$\partial_z\phi_{\mathbf{p}}(z) = \frac{i}{\hbar}\kappa_{\mathbf{p}}(z)\phi_{\mathbf{p}}(z)\,, \tag{357}$$

thus the amplitude becomes

$$\begin{aligned}\mathcal{A}_z &= \frac{e}{2p_z}\int dz\ \left(\kappa_{\mathbf{p}} + \kappa_{\mathbf{p}'}\right)\phi^*_{\mathbf{p}'}\phi_{\mathbf{p}} e^{-ik_z z} \\ &= e\int dz\, e^{i(k_0 t - k_x x - k_y y - k_z z)} \\ &= e\int d\xi\,\frac{dz}{d\xi}e^{ik\xi}\,. \end{aligned} \tag{358}$$

Raising the indices gives

$$\mathcal{A}^\mu = -e\int d\xi\,\frac{dx^\mu}{d\xi}e^{ik\xi}\,. \tag{359}$$

This is the same expression as (344) which we obtained for the t-dependent potential. Given the symmetry between the spatial components we can thus use this amplitude to calculate the position shift for a potential dependent on a single space-time coordinate. In the expression for the amplitude x^μ is the classical trajectory of a particle with final momentum p that passes through $(t, \mathbf{x}) = (0, \mathbf{0})$. This emission amplitude.[13] is identical with that for a classical point charge passing through $(t, \mathbf{x}) = (0, \mathbf{0})$

3.3. Cut-off. The expression for the emission amplitude (359) is currently ill-defined because the integrand does not tend to zero as $\xi \to \pm\infty$. To counter this pathology, we introduce a smooth cut-off function $\chi(\xi)$ which has the properties:

- $\chi(\xi)$ takes the value 1 whilst the acceleration is non-zero.
- $\lim_{\xi\to\pm\infty}\chi(\xi) = 0$.

[13] which is already under the $\hbar \to 0$ limit,

Also, we take $\chi(\xi)$ to be a member of a family of such cut-off functions such that we can take the limit $\chi \to 1$, with the property

- $\int_{-\infty}^{+\infty} [\chi'(\xi)]^2 d\xi \to 0$.

The cut-off version of the emission amplitude is thus written

$$\mathcal{A}^\mu(\mathbf{p}, \mathbf{k}) = -e \int_{-\infty}^{+\infty} d\xi \, \frac{dx^\mu}{d\xi} \chi(\xi) \, e^{ik\xi} \,, \tag{360}$$

and is now well-defined. We shall make use of both this expression for $\mathcal{A}^\mu$ along with the result of integrating (360) by parts

$$\mathcal{A}^\mu(\mathbf{p}, \mathbf{k}) = -\frac{ie}{k} \int_{-\infty}^{+\infty} d\xi \, \left[\frac{d^2 x^\mu}{d\xi^2} + \frac{dx^\mu}{d\xi} \chi'(\xi) \right] e^{ik\xi} \,, \tag{361}$$

where we have used the condition that $\chi(\xi) = 1$ if $d^2 x^\mu / d\xi^2 \neq 0$.

3.4. Larmor Formula. The reader may recall that when calculating the contributions to the position shift we included the term δx_{Q2}, defined in (320)

$$\delta x^i_{\mathrm{Q2}} = -\frac{\langle x^i(0) \rangle_{\mathrm{i}}}{\hbar} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \mathcal{A}^{\mu *}(\mathbf{p}, \mathbf{k}) \mathcal{A}_\mu(\mathbf{p}, \mathbf{k}) \left(\frac{\partial p_i}{\partial P_i} - 1 \right) \,, \tag{320}$$

which was evaluated as zero due to the arrangement in the model that the control particle passes through the spacetime origin. It was also stated at the time that this contribution is in fact that which describes the additional shift produced should the control particle not be at spatial origin at $t = 0$. Whilst the truth of this statement does not, due to the model, affect the results we wish to obtain, it is nonetheless worth briefly taking an aside to consider. In the chapter on the classical position shift, we considered the effect of such a change in the point of measurement whilst analysing the linear acceleration due to a space-dependent potential. We refer the reader to the results (228) and (229), where we found that the extra contribution to the position shift was given by

$$\delta z_{\mathrm{extra}} = -\frac{z_0}{m \gamma_0^3 v_0^2} E_{\mathrm{em}} \,, \tag{228}$$

where $z_0 \neq 0$ is the position of the particle at $t = 0$ and v_0 is its final speed. This contribution is written in terms of the energy E_{em} emitted as radiation

$$E_{\text{em}} = \frac{2\alpha_c}{3} \int_{-\infty}^{0} (\gamma^3 \ddot{z})^2 \, dt \,, \tag{229}$$

which we noted is the relativistic Larmor formula for one-dimensional motion.

Let us use the newly derived emission amplitude to calculate δx_{Q2} for this case of linear acceleration. Choosing the direction z as with the classical formulae above, we have $i = 3$ and

$$\delta z_{\text{Q2}} = -\frac{z_0}{\hbar} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \mathcal{A}^{\mu *}(\mathbf{p}, \mathbf{k}) \mathcal{A}_\mu(\mathbf{p}, \mathbf{k}) \left(\frac{dp}{dP} - 1 \right) \,, \tag{362}$$

where we use $p = p_z$ and $P = P_z$ for simplicity. Firstly, we need to find an expression for dp/dP in terms of p and k to order $\hbar$. The energy conservation equation $p_0 - P_0 = \hbar k$ gives a one-to-one relation between p and P for a given k after letting the transverse momenta $\mathbf{P}_\perp^2 = \mathbf{p}_\perp^2 = 0$, because these are of order $\hbar^2$ here. We then write

$$P_0 = \left(P^2 + m^2 \right)^{1/2} = p_0 - \hbar k \,, \tag{363}$$

and thus

$$\begin{aligned} P^2 &= p_0^2 - 2\hbar k p_0 - m^2 + \mathcal{O}(\hbar^2) \\ &= p^2 - 2\hbar k p_0 + \mathcal{O}(\hbar^2) \,. \end{aligned} \tag{364}$$

Solving for P and expanding the resultant square root gives to order $\hbar$

$$P = p - \frac{p_0}{p} \hbar k = p - \left(1 + \frac{m^2}{p^2} \right)^{1/2} \hbar k \,. \tag{365}$$

This leads us, to order $\hbar$, to

$$\frac{dP}{dp} = 1 + \frac{m^2}{p^2 p_0} \hbar k \,, \tag{366}$$

and finally,

$$\frac{dp}{dP} = 1 - \frac{m^2}{p^2 p_0} \hbar k \,. \tag{367}$$

By using this formula in Eq. (362) we obtain

$$\delta z_{\mathrm{Q2}} = -\frac{m^2 z_0}{p^2 p_0}\mathcal{E}_{\mathrm{em}}\,, \tag{368}$$

where

$$\mathcal{E}_{\mathrm{em}} \equiv -\int \frac{d^3\mathbf{k}}{2k(2\pi)^3} k\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k})\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\,, \tag{369}$$

is the expectation value of the energy emitted as radiation. By comparing (368) with the classical expression (228), it can be seen that the equality $\delta z_{\mathrm{Q2}} = \delta z_{\mathrm{extra}}$ will hold if we can show that

$$\mathcal{E}_{\mathrm{em}} = \frac{2\alpha_c}{3}\int_{-\infty}^{0}(\gamma^3\ddot{z})^2\,dt\,, \tag{370}$$

which is or course identical to the relativistic generalization of the *classical* Larmor formula. For this purpose, we now use the emission amplitude. It is convenient to use the form (361) of the emission amplitude, which was produced by integration by parts. Substituting this expression into the expectation value (369) we obtain

$$\begin{aligned}\mathcal{E}_{\mathrm{em}} = -e^2 \int d\Omega \int_0^\infty \frac{dk}{16\pi^3}\int_{-\infty}^{+\infty} d\xi' \int_{-\infty}^{+\infty} d\xi \\ \times\left[\frac{d^2x^\mu}{d\xi'^2} + \frac{dx^\mu}{d\xi'}\chi'(\xi')\right]\left[\frac{d^2x_\mu}{d\xi^2} + \frac{dx_\mu}{d\xi}\chi'(\xi)\right]e^{ik(\xi-\xi')}\,,\end{aligned} \tag{371}$$

where $d\Omega$ is the solid angle in the $\mathbf{k}$-space and now $\xi = t - z\cos\theta$. We may extend the integration range for k from $[0,+\infty)$ to $(-\infty,+\infty)$ and divide by two. We can then integrate over k to produce the delta function and integrate out the variable ξ' to find

$$\mathcal{E}_{\mathrm{em}} = -\frac{e^2}{16\pi^2}\int d\Omega \int_{-\infty}^{+\infty} d\xi \left\{\frac{d^2x_\mu}{d\xi^2}\frac{d^2x^\mu}{d\xi^2} + \frac{dx_\mu}{d\xi}\frac{dx^\mu}{d\xi}\left[\chi'(\xi)\right]^2\right\}. \tag{372}$$

Noting that $(dx_\mu/d\xi)(dx^\mu/d\xi)$ is bounded, the second term tends to zero in the limit $\chi(\xi) \to 1$ due to the requirement

$$\int_{-\infty}^{+\infty}[\chi'(\xi)]^2 d\xi \to 0\,. \tag{373}$$

Hence, we have in this limit

$$\mathcal{E}_{\mathrm{em}} = -\frac{\alpha_c}{4\pi}\int_{-\infty}^{+\infty} d\xi \int d\Omega\, \frac{d^2x_\mu}{d\xi^2}\frac{d^2x^\mu}{d\xi^2}\,, \tag{374}$$

where $\alpha_c \equiv e^2/4\pi$ as before. Now, one can readily show that

$$\frac{d^2x^\mu}{d\xi^2} = \left(\frac{dt}{d\xi}\right)^3 \left[\frac{d\xi}{dt}\frac{d^2x^\mu}{dt^2} - \frac{d^2\xi}{dt^2}\frac{dx^\mu}{dt}\right]. \tag{375}$$

By substituting $d\xi/dt = 1 - \dot{z}\cos\theta$ we find

$$\frac{d^2z}{d\xi^2} = \frac{\ddot{z}}{(1-\dot{z}\cos\theta)^3}, \tag{376}$$

$$\frac{d^2t}{d\xi^2} = \frac{d^2z}{d\xi^2}\cos\theta, \tag{377}$$

and hence

$$\frac{d^2x^\mu}{d\xi^2}\frac{d^2x_\mu}{d\xi^2} = -\frac{\ddot{z}^2\sin^2\theta}{(1-\dot{z}\cos\theta)^5}\frac{dt}{d\xi}. \tag{378}$$

By substituting this formula in (374) we obtain

$$\begin{aligned}\mathcal{E}_{\rm em} &= \frac{\alpha_c}{4\pi}\int_{-\infty}^{\infty} d\xi \int d\Omega \frac{\ddot{z}^2\sin^2\theta}{(1-\dot{z}\cos\theta)^5}\frac{dt}{d\xi} \\ &= \frac{\alpha_c}{2}\int_{-\infty}^{\infty} dt \int_0^\pi d\theta \frac{\ddot{z}^2\sin^3\theta}{(1-\dot{z}\cos\theta)^5} \\ &= \frac{\alpha_c}{2}\int_{-\infty}^{\infty} dt\ddot{z}^2\frac{4}{3}\frac{1}{(1-\dot{z}^2)^3} \\ &= \frac{2\alpha_c}{3}\int_{-\infty}^{0} dt(\gamma^3\ddot{z})^2.\end{aligned} \tag{379}$$

The limits for the last line were changed from $(-\infty,\infty)$ to $(-\infty,0)$ by virtue of the fact that $\ddot{z}=0$ for $t\geq 0$. Consequently, we have

$$\delta z_{\rm Q2} = \delta z_{\rm extra}, \tag{380}$$

as required.

3.5. Position shift. We now use the emission amplitude expressions to find the emission contribution to the quantum position shift. Recall that this was given in (325) as

$$\delta x^i_{\rm em} = -\frac{i}{2}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial}_{p_i} \mathcal{A}_\mu(\mathbf{p},\mathbf{k}). \tag{325}$$

In the product $\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k})\partial_{p_i}\mathcal{A}_\mu(\mathbf{p},\mathbf{k})$, we shall use the expression (361) for $\mathcal{A}^{\mu *}$ and (360) for $\mathcal{A}_\mu$ (and vice-versa for the conjugate).[14] This leads to

$$\begin{aligned}\delta x^i_{\rm em} = -\frac{e^2}{2}\int d\Omega \int_0^\infty \frac{k^2 dk}{16\pi^3 k^2}\int_{-\infty}^{+\infty} d\xi' \int_{-\infty}^{+\infty} d\xi \\ \times \left\{\left[\frac{d^2x^\mu}{d(\xi')^2} + \frac{dx^\mu}{d(\xi')}\chi'(\xi')\right]\frac{\partial}{\partial p_i}\left(\frac{dx_\mu}{d\xi}\right)\chi(\xi) + (\xi \leftrightarrow \xi')\right\} e^{ik(\xi-\xi')}\,,\end{aligned} \tag{381}$$

where $d\Omega$ is in angular part of the $\mathbf{k}$ integration in spherical polar coordinates. Due to the symmetry in the integration, we make the swap back $\xi \leftrightarrow \xi'$ to the second term. However this will change the exponential, producing the same overall effect as the transformation $k \to -k$ on the first term, viz

$$\begin{aligned}\delta x^i_{\rm em} = -\frac{e^2}{2}\int d\Omega \int_0^\infty \frac{k^2 dk}{16\pi^3 k^2}\int_{-\infty}^{+\infty} d\xi' \int_{-\infty}^{+\infty} d\xi \\ \times \left[\frac{d^2x^\mu}{d(\xi')^2} + \frac{dx^\mu}{d(\xi')}\chi'(\xi')\right]\frac{\partial}{\partial p_i}\left(\frac{dx_\mu}{d\xi}\right)\chi(\xi)\left[e^{ik(\xi-\xi')} + e^{-ik(\xi-\xi')}\right]\,.\end{aligned} \tag{382}$$

This expression makes it clearer that the second term is the conjugate of the first, as is known from (325). Making the substitution of integration variables $k \to -k$ for the second term, we see that it is integrand is identical to the first, but the integration range is now $(-\infty, 0)$. We may thus rewrite

$$\begin{aligned}\delta x^i_{\rm em} = -\frac{e^2}{2}\int d\Omega \int_{-\infty}^\infty \frac{k^2 dk}{16\pi^3 k^2}\int_{-\infty}^{+\infty} d\xi' \int_{-\infty}^{+\infty} d\xi \\ \times \left[\frac{d^2x^\mu}{d(\xi')^2} + \frac{dx^\mu}{d(\xi')}\chi'(\xi')\right]\frac{\partial}{\partial p_i}\left(\frac{dx_\mu}{d\xi}\right)\chi(\xi)e^{ik(\xi-\xi')}\,,\end{aligned} \tag{383}$$

where the k integration is over the full range $(-\infty, \infty)$. We can now integrate over k to produce the delta function $2\pi\delta(\xi - \xi')$ and consequently integrate

[14]Equation (360) was where we introduced the cut-off in $\mathcal{A}^\mu$ and (361) was the result of integrating by parts.

out ξ' to give

$$\delta x^i_{\rm em} = -\frac{e^2}{16\pi^2}\int d\Omega \int_{-\infty}^{+\infty} d\xi \left[\frac{d^2x^\mu}{d\xi^2} + \frac{dx^\mu}{d\xi}\chi'(\xi)\right]\frac{\partial}{\partial p_i}\left(\frac{dx_\mu}{d\xi}\right)\chi(\xi)$$
$$= -\frac{e^2}{16\pi^2}\int d\Omega \int d\xi \left[\frac{d^2x^\mu}{d\xi^2}\frac{\partial}{\partial p_i}\left(\frac{dx^\mu}{d\xi}\right) + \frac{1}{4}\frac{\partial}{\partial p_i}\left(\frac{dx^\mu}{d\xi}\frac{dx_\mu}{d\xi}\right)\frac{d}{d\xi}(\chi(\xi))^2\right], \tag{384}$$

where, whilst combining terms we have recalled the property $\chi(\xi) = 1$ if $d^2x^\mu/d\xi^2 \neq 0$. Noting that

$$\frac{dx^\mu}{d\xi}\frac{dx_\mu}{d\xi} = \left(\frac{d\tau}{d\xi}\right)^2\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = \left(\frac{d\tau}{d\xi}\right)^2, \tag{385}$$

we find that the second term in (384) above is proportional to the integral

$$I \equiv \int d\Omega \int_{-\infty}^{+\infty} d\xi \frac{d}{d\xi}\left\{\frac{\partial}{\partial p_i}\left(\frac{d\tau}{d\xi}\right)^2\right\}(\chi(\xi))^2. \tag{386}$$

We shall now show that this integral is in fact zero: Owing to the fact that $\partial/\partial p_i$ is taken with ξ fixed, the ξ-derivative and the p_i-derivative commute. Hence

$$I = \int d\Omega \int_{-\infty}^{+\infty} d\xi \frac{\partial}{\partial p_i}\left[\frac{d}{d\xi}\left(\frac{d\tau}{d\xi}\right)^2\right](\chi(\xi))^2. \tag{387}$$

This expression is simplified by the observation that the quantity inside the square brackets is nonzero only if the acceleration is nonzero, or correspondingly when $\chi(\xi) = 1$. Relocating the p_i differentiation outside the integration in I we have

$$I = \frac{\partial}{\partial p_i}\int d\Omega \int_{-\infty}^{+\infty} d\xi \frac{d}{d\xi}\left(\frac{d\tau}{d\xi}\right)^2$$
$$= \frac{\partial}{\partial p_i}\int d\Omega \left[\left(\frac{d\tau}{d\xi}\right)^2\right]_{\xi=-\infty}^{\xi=+\infty}. \tag{388}$$

Now, the quantity to be differentiated is

$$\int d\Omega \left(\frac{d\tau}{d\xi}\right)^2 = 2\pi\int_0^\pi d\theta \sin\theta \left(\frac{dt}{d\tau} - \frac{dz}{d\tau}\cos\theta\right)^{-2}$$
$$= \frac{4\pi}{(dt/d\tau)^2 - (dz/d\tau)^2} = 4\pi, \tag{389}$$

i.e. a constant, ergo the integral $I = 0$ as stated. If one recalls the definition $\alpha_c = e^2/4\pi$, then the emission contribution to the quantum position shift can now be written in a fairly compact form

$$\delta x^i_{\text{em}} = -\frac{\alpha_c}{4\pi}\int d\Omega \int d\xi \frac{d^2x^\mu}{d\xi^2}\frac{\partial}{\partial p_i}\left(\frac{dx_\mu}{d\xi}\right) . \tag{390}$$

We additionally note that this expression is now independent of the cut-off function.

Whilst (390) is a fairly simple compact expression, it is still one in a somewhat different form to that of the similarly compact classical position shift as given in (279), which is written in terms of t rather than ξ. We thus need to eliminate the variable ξ using its definition $\xi = t - \mathbf{n}\cdot\mathbf{x}$. [15] Before tackling the second derivative term, we may again interchange the order of the p_i and ξ derivatives and change integration variables from ξ to t:

$$\delta x^i_{\text{em}} = -\frac{\alpha_c}{4\pi}\int d\Omega \int dt \frac{d^2x_\mu}{d\xi^2}\frac{d}{dt}\left(\frac{\partial x^\mu}{\partial p_i}\right)_\xi . \tag{391}$$

We remind the reader that in the above expression $d\Omega$ is the solid angle in the wave-number space $\mathbf{k}$ of the emitted photon. We have additionally placed the subscript ξ on the final partial derivative to emphasize that this variable is held fixed, which will be important when it is evaluated. Furthermore we remind the reader that the momentum p_i is the *final* momentum in the measurement region $\mathcal{M}_+$. Consequently we would write the velocity $dx^\mu/dt = (\partial x^\mu/\partial t)_{p_i}$.

Proceeding with the evaluation of (391) via the elimination of ξ, one can readily write $d^2x^\mu/d\xi^2$ in terms of t-derivatives by using $d/d\xi = (1 - n^i\dot{x}^i)^{-1}d/dt$ as follows:

$$\frac{d^2x^\mu}{d\xi^2} = \dot{\xi}^{-3}\left[\left(1 - n^i\dot{x}^i\right)\ddot{x}^\mu + n^i\ddot{x}^i\dot{x}^\mu\right] , \tag{392}$$

where $\dot{\xi} = 1 - n^i\dot{x}^i$. Here and in the rest of this section, Latin indices, which we recall take the spatial values 1 to 3, are summed over when repeated. The

[15] Recall that we defined $\mathbf{n} = \mathbf{k}/k$.

time and space components of (392) can separately be given as

$$\frac{d^2t}{d\xi^2} = \dot{\xi}^{-3}\, n^i \ddot{x}^i\,, \tag{393}$$

$$\frac{d^2x^j}{d\xi^2} = \dot{\xi}^{-3}\left[\left(1 - n^i\dot{x}^i\right)\ddot{x}^j + n^i\ddot{x}^i\dot{x}^j\right]\,. \tag{394}$$

Next we express $(\partial x^\mu/\partial p_i)_\xi$ in (391) in the form involving t rather than ξ as follows. Note first

$$dx^\mu = \frac{dx^\mu}{dt}dt + \left(\frac{\partial x^\mu}{\partial p_i}\right)_t dp_i\,. \tag{395}$$

The zeroth component of this equation is in fact trivial because $(\partial t/\partial p_i)_t = 0$. By substituting $dt = d\xi + n^k dx^k$ in this equation with $\mu = j$ we have

$$dx^i = \frac{dx^i}{dt}d\xi + \frac{dx^i}{dt}n^j dx^j + \left(\frac{\partial x^i}{\partial p^j}\right)_t dp^j\,. \tag{396}$$

We can solve (396) for dx^i by first observing that by contracting both sides of (396) with n^j we can solve to find an expression for $n^k dx^k$, viz

$$n^k dx^k = \frac{n^j v^j}{1 - n^j v^j}d\xi + \frac{n^k}{1 - n^j v^j}\left(\frac{\partial x^k}{\partial p^j}\right)_t dp^j\,, \tag{397}$$

where $v^i \equiv \dot{x}^i = dx^i/dt$. By substituting this back into (396), we solve for dx^i

$$dx^i = \frac{v^i}{1 - n^j v^j}d\xi + \frac{\left[\delta^{ik}(1 - n^l v^l) + n^k v^i\right]}{1 - n^l v^l}\left(\frac{\partial x^k}{\partial p^j}\right)_t dp^j\,. \tag{398}$$

Hence

$$\left(\frac{\partial x^i}{\partial p_j}\right)_\xi = \frac{\left[\delta^{ik}(1 - n^l v^l) + n^k v^i\right]}{1 - n^l v^l}\left(\frac{\partial x^k}{\partial p_j}\right)_t\,. \tag{399}$$

With ξ fixed we have $dt - n^i dx^i = d\xi = 0$. Thus,

$$\begin{aligned}\left(\frac{\partial t}{\partial p_j}\right)_\xi &= n^i\left(\frac{\partial x^i}{\partial p_j}\right)_\xi \\ &= \frac{n^i}{1 - n^l v^l}\left(\frac{\partial x^i}{\partial p_j}\right)_t\,.\end{aligned} \tag{400}$$

By substituting the pairs (393), (394) and (399), (400) in the position shift (391) we find, after a straightforward albeit lengthy amount of rearranging,

$$\delta x^i_Q = -\frac{\alpha_c}{4\pi}\int dt\left\{\left[I_2^{kj}\gamma^{-2}a^j - I_0 a^k - I_1^j a^j v^k - I_1^k(\mathbf{a}\cdot\mathbf{v})\right]\frac{d}{dt}\left(\frac{\partial x^k}{\partial p^i}\right)_t \right.$$
$$\left. + \left[I_3^{kjl}\gamma^{-2}a^j a^l - 2I_2^{kj}a^j(\mathbf{a}\cdot\mathbf{v}) - I_1^k\mathbf{a}^2\right]\left(\frac{\partial x^k}{\partial p^i}\right)_t\right\}, \quad (401)$$

where

$$I_0 \equiv \int d\Omega\frac{1}{\dot{\xi}^2} = 4\pi\gamma^2\,, \quad (402)$$

$$I_1^i \equiv \int d\Omega\frac{n^i}{\dot{\xi}^3} = 4\pi\gamma^4 v^i\,, \quad (403)$$

$$I_2^{ij} \equiv \int d\Omega\frac{n^i n^j}{\dot{\xi}^4} = \frac{16}{3}\pi\gamma^6 v^i v^j + \frac{4}{3}\pi\gamma^4\delta^{ij}\,, \quad (404)$$

$$I_3^{ijk} \equiv \int d\Omega\frac{n^i n^j n^k}{\dot{\xi}^5} = 8\pi\gamma^8 v^i v^j v^k + \frac{4}{3}\pi\gamma^6\left(v^i\delta^{jk} + v^j\delta^{ik} + v^k\delta^{ij}\right)\,. \quad (405)$$

Evaluation of these solid-angle integrals is facilitated by noting that the last three integrals are proportional to partial derivatives of I_0 with respect to v^i, explicitly

$$I_n^{i_1\ldots i_n} = \frac{1}{(n+1)!}\frac{\partial}{\partial v^{i_n}}I_{n-1} \quad n = 1, 2, 3\,. \quad (406)$$

Substitution of (402)–(405) in (401) yields

$$\delta x^i_{\rm em} = \frac{2\alpha_c}{3}\int dt\left\{\left[\gamma^4(\mathbf{a}\cdot\mathbf{v})v^k + \gamma^2 a^k\right]\frac{d}{dt}\left(\frac{\partial x^k}{\partial p^i}\right)_t \right.$$
$$\left. + \left[\gamma^6(\mathbf{a}\cdot\mathbf{v})^2 v^k + \gamma^4\mathbf{a}^2 v^k\right]\left(\frac{\partial x^k}{\partial p^i}\right)_t\right\}, \quad (407)$$

which, by using the fact that $\mathbf{a}(t) \neq 0$ only for $t < 0$ and for a finite interval of time to integrate the first term by parts, becomes

$$\delta x^i_{\rm em} = -\frac{2\alpha_c}{3}\int dt\left\{\frac{d}{dt}\left[\gamma^4(\mathbf{a}\cdot\mathbf{v})v^k + \gamma^2 a^k\right]\right.$$
$$\left. -\gamma^6(\mathbf{a}\cdot\mathbf{v})^2 v^k - \gamma^4\mathbf{a}^2 v^k\right\}\left(\frac{\partial x^k}{\partial p^i}\right)_t\,. \quad (408)$$

Comparison with the expression for the Lorentz-Dirac force in (277) demonstrates that we have

$$\delta x^i_{\mathrm{em}} = -\int_{-\infty}^{0} dt\, \mathcal{F}^j_{\mathrm{LD}} \left(\frac{\partial x^j}{\partial p^i}\right)_t . \tag{409}$$

We recognize this as equal to the classical position shift (279).

4. Forward Scattering

The forward scattering contribution to the position shift was shown in (324) to be equal to

$$\delta x^i_{\mathrm{for}} = -\hbar \partial_{p_i} \Re \mathcal{F}(\mathbf{p}) . \tag{324}$$

Considering this expression there are two points to bear in mind when calculating the forward scattering amplitude. Firstly, we note that we only require the real part of the amplitude. Recall that the imaginary part was canceled via its relation to the emission probability. Secondly, we note that we have an additional $\hbar$ factor multiplying $\mathcal{F}(\mathbf{p})$. This is not contrary to the fact that (324) is in the $\hbar \to 0$ limit, as we shall shortly see that $\mathcal{F}$ is of order $\hbar^{-2}$. We shall thus actually need the first *two* orders, $\hbar^{-2}$ and $\hbar^{-1}$, in $\mathcal{F}$. With the extra $\hbar$ factor, these orders will contribute at orders $\hbar^{-1}$ and $\hbar^0$. In taking the $\hbar \to 0$ limit, one would wish the former to be zero or canceled. Additionally, should the latter be non-zero, then we would have an additional position shift contribution in the $\hbar \to 0$ limit. As the emission position shift has already been shown to give the classical position shift expression, such an additional contribution would present a quantum correction to the classical theory.

In this section we shall in fact show that the overall contribution from the forward scattering towards the position shift is zero when we take the renormalisation of the mass into account. This is in fact in keeping with the classical description of radiation reaction, although with some subtle differences, and we shall return to this later. We proceed with the calculation of the forward scattering amplitude and shall then continue to calculate the contribution to

the renormalised forward scattering made by the mass counterterm. The forward scattering amplitude was first introduced, and consequently given its definition, by its presence as part of the amplitude of the zero-photon sector of the final state, viz

$$|f\rangle_{\text{for}} = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3\sqrt{2p_0}} \left[1 + i\mathcal{F}(\mathbf{p})\right] f(\mathbf{p}) A^\dagger(\mathbf{p})|0\rangle \,. \tag{288}$$

If we equate this expression with the state evolution in time-dependent perturbation theory, then to $\mathcal{O}(e^2)$ we have the following zero photon sectors:

$$\begin{aligned} i\mathcal{F}(\mathbf{p})A^\dagger(\mathbf{p})|0\rangle = & -\frac{i}{\hbar}\int d^4x \mathcal{H}_I A^\dagger(\mathbf{p})|0\rangle\Big|_{\text{zero}-\text{photon}} \\ & +\frac{1}{2}\left(\frac{-i}{\hbar}\right)^2 \int d^4x d^4x' T\left[\mathcal{H}_I(x')\mathcal{H}_I(x)\right] A^\dagger(\mathbf{p})|0\rangle\Big|_{\text{zero}-\text{photon}} \,, \end{aligned} \tag{410}$$

where $T[\ldots]$ represents the usual time ordering and zero − photon indicates that we require only the zero photon terms. We note that we need both the first and second orders in the interaction Hamiltonian for this order of e^2. Recall that $\mathcal{H}_I$ was given by

$$\mathcal{H}_I(x) = \frac{ie}{\hbar} A_\mu{:}\left[\varphi^\dagger D^\mu\varphi - (D^\mu\varphi)^\dagger\varphi\right]{:} + \frac{e^2}{\hbar^2}\sum_{i=1}^{3} A_i A_i{:}\varphi^\dagger\varphi{:}\,. \tag{85}$$

We see here that the first term in (85) contributes at second order in $\mathcal{H}_I$ for (410), whilst the second term in (85) contributes at first order in $\mathcal{H}_I$. Operating on both sides of (410) with the bra-state $\langle 0|A(\mathbf{p}')$, which invokes the zero photon condition, and using the commutation relations for the resulting inner product, we may rearrange the result to produce the forward scattering:

$$\begin{aligned} \mathcal{F}(\mathbf{p}) = & -\frac{1}{\hbar}\int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} d^4x \langle 0|A(\mathbf{p}')\mathcal{H}_I(x)A^\dagger(\mathbf{p})|0\rangle \\ & + \frac{i}{2\hbar^2}\int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} d^4x d^4x' \langle 0|A(\mathbf{p}')T\left[\mathcal{H}_I(x')\mathcal{H}_I(x)\right] A^\dagger(\mathbf{p})|0\rangle \,. \end{aligned} \tag{411}$$

The portion of the above expression at first order in $\mathcal{H}_I$ we label $\mathcal{F}_1$ and correspondingly name the remainder $\mathcal{F}_2$. [16]

[16] In [**7**] this notation was the reverse of that given here.

4.1. $\mathcal{F}_1$. The calculation of this first order $\mathcal{F}_1$ part is fairly straightforward and we present it here. Substituting the interaction Hamiltonian into the definition of $\mathcal{F}_1$, we have

$$\begin{aligned}\mathcal{F}_1(\mathbf{p}) &= -\frac{1}{\hbar}\frac{e^2}{\hbar^2}\int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} d^4x \sum_{i=1}^{3} \langle 0|A(\mathbf{p}')A_i(x)A_i(x){:}\varphi^\dagger(x)\varphi(x){:}A^\dagger(\mathbf{p})|0\rangle \\ &= -\frac{e^2}{\hbar^3}\int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} d^4x \sum_{i=1}^{3} \langle 0|A_iA_i|0\rangle \langle 0|A(\mathbf{p}'){:}\varphi^\dagger\varphi{:}A^\dagger(\mathbf{p})|0\rangle \,. \qquad (412)\end{aligned}$$

Let us analyse the two inner products in turn, using the appropriate commutation relations. For the first product, involving the electromagnetic field (which is not normal ordered) only the annihilation operator from the first field and the creation operator from the second field will give non-zero results. Thus

$$\begin{aligned}\langle 0|A_iA_i|0\rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\frac{d^3\mathbf{k}'}{(2\pi)^3 2k_0'} \sum_{i=1}^{3} \langle 0|a_i(\mathbf{k})a_i^\dagger(\mathbf{k}')|0\rangle e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\frac{d^3\mathbf{k}'}{(2\pi)^3 2k_0'} \sum_{i=1}^{3} \left(-\hbar g_{ii} 2k_0'(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')\right) e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\ &= 3\hbar \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\,, \qquad (413)\end{aligned}$$

where we recall that we are using the time-like sign convention $(+---)$. The second product, involving the scalar particle fields is

$$\begin{aligned}\langle 0|A(\mathbf{p}'){:}\varphi^\dagger\varphi{:}A^\dagger(\mathbf{p})|0\rangle = \hbar^2 &\int \frac{d^3\mathbf{p}^{(\mathbf{2})}}{(2\pi\hbar)^3 2p_0^{(2)}} \int \frac{d^3\mathbf{p}^{(\mathbf{3})}}{(2\pi\hbar)^3 2p_0^{(3)}} \\ &\times \langle 0|A(\mathbf{p}')A^\dagger(\mathbf{p}^{(\mathbf{2})})A(\mathbf{p}^{(\mathbf{3})})A^\dagger(\mathbf{p})|0\rangle \Phi^*_{\mathbf{p}^{(\mathbf{2})}}(x)\Phi_{\mathbf{p}^{(\mathbf{3})}}(x)\,. \qquad (414)\end{aligned}$$

The remaining operators from the normal ordered fields are annihilated by the vacuum (after normal ordering). We also recall the overall prefactor of $\hbar$ in the expression for φ (see (71)) thus leading to the $\hbar^2$ above. Using the commutation relations we have

$$\langle 0|A(\mathbf{p}'){:}\varphi^\dagger\varphi{:}A^\dagger(\mathbf{p})|0\rangle = \hbar^2 \Phi^*_{\mathbf{p}'}(x)\Phi_{\mathbf{p}}(x)\,, \qquad (415)$$

and consequently, combining with (413),

$$\begin{aligned}\mathcal{F}_1(\mathbf{p}) &= -3e^2 \int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} d^4x \Phi^*_{\mathbf{p}'}(x)\Phi_{\mathbf{p}}(x) \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} \\ &= -3e^2 \int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} d^4x \phi^*_{\mathbf{p}'}(t)\phi_{\mathbf{p}}(t) e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}/\hbar} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} \\ &= -\frac{3e^2}{2p_0} \int dt |\phi_{\mathbf{p}}(t)|^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} \,. \end{aligned} \tag{416}$$

The use of k in the integration hides the $\hbar$ dependence of this term. Changing the integration variable to the photon momentum $K = \hbar k$, we use $d^3\mathbf{k}/k = \hbar^{-2} d^3\mathbf{K}/K$. [17] Thus

$$\mathcal{F}_1(\mathbf{p}) = -\frac{3e^2}{2\hbar^2 p_0} \int dt |\phi_{\mathbf{p}}(t)|^2 \int \frac{d^3\mathbf{K}}{(2\pi)^3 2K} \,. \tag{417}$$

4.2. $\mathcal{F}_2$. The portion of the forward scattering at second order in the Hamiltonian forms the bulk of the calculation and is more complicated than the above first order term. We start from

$$\mathcal{F}_2(\mathbf{p}) = +\frac{i}{2\hbar^2} \int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} d^4x d^4x' \langle 0| A(\mathbf{p}') T\left[\mathcal{H}_I(x')\mathcal{H}_I(x)\right] A^\dagger(\mathbf{p})|0\rangle \,. \tag{418}$$

That part of $\mathcal{H}_I$ which is relevant[18] is

$$\frac{ie}{\hbar} A_\mu \colon\! \left[\varphi^\dagger D^\mu \varphi - (D^\mu\varphi)^\dagger \varphi\right] \colon\!= \frac{ie}{\hbar} A_\mu \colon\! \left[\varphi^\dagger \overleftrightarrow{D}^\mu \varphi\right] \colon\!, \tag{419}$$

where we have used the more compact notation $\overleftrightarrow{D}^\mu = \overrightarrow{D}^\mu - \overleftarrow{D}^{\dagger\mu}$. The inner product in (418) can be separated into the electromagnetic and scalar field parts:

$$\begin{aligned} &\langle 0| A(\mathbf{p}') T\left[\mathcal{H}_I(x')\mathcal{H}_I(x)\right] A^\dagger(\mathbf{p})|0\rangle \\ &= \frac{-e^2}{\hbar^2} \langle 0| A(\mathbf{p}') T\left[A_{\mu'}(x') \colon\! \varphi^\dagger(x') \overleftrightarrow{D}^{\mu'} \varphi(x') \colon\! A_\nu(x) \colon\! \varphi^\dagger(x) \overleftrightarrow{D}^\nu \varphi(x) \colon \right] A^\dagger(\mathbf{p})|0\rangle \\ &= \frac{-e^2}{\hbar^2} \langle 0| A(\mathbf{p}') T\left[\colon\! \varphi^\dagger(x') \overleftrightarrow{D}^{\mu'} \varphi(x') \colon\colon\! \varphi^\dagger(x) \overleftrightarrow{D}^\nu \varphi(x) \colon \right] A^\dagger(\mathbf{p})|0\rangle \\ &\quad \times \langle 0| T\left[A_{\mu'}(x') A_\nu(x)\right]|0\rangle \,. \end{aligned} \tag{420}$$

[17] Where $K = |\mathbf{K}|$ analogously to k.

[18] Recall that we only require terms up to $\mathcal{O}(e^2)$.

The second inner product is simply the photon propagator, the expression for which is

$$\begin{aligned}
&\langle 0|T\left[A_{\mu'}(x')A_\nu(x)\right]|0\rangle \\
&= -\hbar g_{\mu'\nu}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\left[\theta(t-t')e^{-ik\cdot(x-x')}+\theta(t'-t)e^{ik\cdot(x-x')}\right]. \qquad (421)
\end{aligned}$$

The scalar field product is more complicated and the full notation becomes somewhat cumbersome; In full, with the expansions of the fields, we have

$$\begin{aligned}
&\langle 0|A(\mathbf{p}')T\left[:\varphi^\dagger(x')\overleftrightarrow{D}^{\mu'}\varphi(x'):\ :\varphi^\dagger(x)\overleftrightarrow{D}^{\nu}\varphi(x):\right]A^\dagger(\mathbf{p})|0\rangle \\
&= \hbar^4\int \frac{d^3\mathbf{p^{(2)}}}{(2\pi\hbar)^3 2p_0^{(2)}}\frac{d^3\mathbf{p^{(3)}}}{(2\pi\hbar)^3 2p_0^{(3)}}\frac{d^3\mathbf{p^{(4)}}}{(2\pi\hbar)^3 2p_0^{(4)}}\frac{d^3\mathbf{p^{(5)}}}{(2\pi\hbar)^3 2p_0^{(5)}} \\
&T\langle 0|A(\mathbf{p}'):\left(A_2^\dagger\Phi_2^*(x')+B_2\bar{\Phi}_2(x')\right)\overleftrightarrow{D}^{\mu'}\left(A_3\Phi_3(x')+B_3^\dagger\bar{\Phi}_3^*(x')\right): \\
&:\left(A_4^\dagger\Phi_4^*(x)+B_4\bar{\Phi}_4(x)\right)\overleftrightarrow{D}^{\nu}\left(A_5\Phi_5(x)+B_5^\dagger\bar{\Phi}_5^*(x)\right):A^\dagger(\mathbf{p})|0\rangle\,, \qquad (422)
\end{aligned}$$

where the subscript gives the momentum of the operator or mode function, eg $A_j \equiv A(\mathbf{p^{(j)}})$. Considering only the operators momentarily, we have for each interaction Hamiltonian term

$$:\left(A_i^\dagger+B_i\right)\left(A_j+B_j^\dagger\right):=A_i^\dagger A_j+A_i^\dagger B_j^\dagger+B_iA_j+B_j^\dagger B_i\,. \qquad (423)$$

Operating on this combination on the left by $\langle 0|A(\mathbf{p}')$, the second and fourth terms are annihilated by the vacuum. Similarly, when operated on the right by $A^\dagger(\mathbf{p})|0\rangle$ the third and fourth are annihilated. Overall, in terms of just the operators, we then have

$$\langle 0|A(\mathbf{p}')A_2^\dagger A_3A_4^\dagger A_5A^\dagger(\mathbf{p})|0\rangle+\langle 0|A(\mathbf{p}')B_2A_3A_4^\dagger B_5^\dagger A^\dagger(\mathbf{p})|0\rangle\,, \qquad (424)$$

from which we obtain the delta functions

$$\begin{aligned}
&(2\pi\hbar)^9 2^3 p_0^{(2)}p_0^{(4)}p_0^{(5)}\delta^3(\mathbf{p}'-\mathbf{p^{(2)}})\delta^3(\mathbf{p^{(3)}}-\mathbf{p^{(4)}})\delta^3(\mathbf{p}-\mathbf{p^{(5)}}) \\
&+(2\pi\hbar)^9 2^3 p_0^{(2)}p_0^{(3)}p_0^{(4)}\delta^3(\mathbf{p}'-\mathbf{p^{(4)}})\delta^3(\mathbf{p^{(2)}}-\mathbf{p^{(5)}})\delta^3(\mathbf{p}-\mathbf{p^{(3)}})\,, \qquad (425)
\end{aligned}$$

and an additional term representing the vacuum pair creation and annihilation event which we can ignore.[19] Using the appropriate mode functions and integrating out the delta functions, what remains is the following

$$\hbar^4 \int \frac{d^3\mathbf{q}}{(2\pi\hbar)^3 2q_0}$$
$$T\left[\Phi^*_{\mathbf{p'}}(x') \overleftrightarrow{D}^{\mu'} \Phi_{\mathbf{q}}(x')\Phi^*_{\mathbf{q}}(x) \overleftrightarrow{D}^{\nu} \Phi_{\mathbf{p}}(x) + \bar{\Phi}_{\mathbf{q}}(x') \overleftrightarrow{D}^{\mu'} \Phi_{\mathbf{p}}(x')\Phi^*_{\mathbf{p'}}(x) \overleftrightarrow{D}^{\nu} \bar{\Phi}_{\mathbf{q}}(x)\right], \tag{426}$$

where we have used $\mathbf{q}$ for the internal momentum (previously $\mathbf{p^{(3)}}$ in one case and $\mathbf{p^{(5)}}$ in the other).

We can now combine this result with that for the electromagnetic fields.

$$\begin{aligned}
&\mathcal{F}_2(\mathbf{p}) \\
&= \frac{ie^2\hbar}{2} \int d^4x d^4x' \frac{d^3\mathbf{p'}}{2p'_0(2\pi\hbar)^3} \frac{d^3\mathbf{q}}{(2\pi\hbar)^3 2q_0} \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} \\
&\Big\{\theta(t'-t)e^{ik\cdot(x-x')} \\
&\left(\Phi^*_{\mathbf{p'}}(x') \overleftrightarrow{D}'_{\mu} \Phi_{\mathbf{q}}(x')\Phi^*_{\mathbf{q}}(x) \overleftrightarrow{D}^{\mu} \Phi_{\mathbf{p}}(x) + \Phi^*_{\mathbf{p'}}(x) \overleftrightarrow{D}^{\mu} \bar{\Phi}_{\mathbf{q}}(x)\bar{\Phi}_{\mathbf{q}}(x') \overleftrightarrow{D}'_{\mu} \Phi_{\mathbf{p}}(x')\right) \\
&+ \theta(t-t')e^{-ik\cdot(x-x')} \\
&\left(\Phi^*_{\mathbf{p'}}(x) \overleftrightarrow{D}_{\mu} \Phi_{\mathbf{q}}(x)\Phi^*_{\mathbf{q}}(x') \overleftrightarrow{D}'^{\mu} \Phi_{\mathbf{p}}(x') + \Phi^*_{\mathbf{p'}}(x') \overleftrightarrow{D}'^{\mu} \bar{\Phi}_{\mathbf{q}}(x')\bar{\Phi}_{\mathbf{q}}(x) \overleftrightarrow{D}_{\mu} \Phi_{\mathbf{p}}(x)\right)\Big\},
\end{aligned} \tag{427}$$

where we have rearranged the last term in each curved bracket so that it matches the first. The two time ordered terms are then the same under the

[19] The same event was ignored for the initial state calculations in this chapter and explained in the first paragraph at the beginning of section 1

(x, x') integration symmetry. We choose to combine them as follows:

$$\mathcal{F}_2(\mathbf{p}) = ie^2\hbar \int d^4x d^4x' \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} \frac{d^3\mathbf{q}}{(2\pi\hbar)^3 2q_0} \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}$$
$$\Big\{\theta(t-t')\Phi^*_{\mathbf{p}'}(x) \overleftrightarrow{D}_\mu \Phi_{\mathbf{q}}(x)\Phi^*_{\mathbf{q}}(x') \overleftrightarrow{D}'^\mu \Phi_{\mathbf{p}}(x')e^{-ik\cdot(x-x')}$$
$$+\theta(t'-t)\Phi^*_{\mathbf{p}'}(x) \overleftrightarrow{D}^\mu \bar{\Phi}_{\mathbf{q}}(x)\bar{\Phi}_{\mathbf{q}}(x') \overleftrightarrow{D}'_\mu \Phi_{\mathbf{p}}(x')e^{ik\cdot(x-x')}\Big\}. \tag{428}$$

These two terms represent the particle and antiparticle loops respectively. We denote them as follows:

$$\mathcal{F}_{2a}(\mathbf{p}) = ie^2\hbar \int d^4x d^4x' \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3 2p_0'} \frac{d^3\mathbf{q}}{(2\pi\hbar)^3 2q_0} \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}$$
$$\Big\{\theta(t-t')\Phi^*_{\mathbf{p}'}(x) \overleftrightarrow{D}_\mu \Phi_{\mathbf{q}}(x)\Phi^*_{\mathbf{q}}(x') \overleftrightarrow{D}'^\mu \Phi_{\mathbf{p}}(x')e^{-ik\cdot(x-x')}\Big\}, \tag{429}$$

and

$$\mathcal{F}_{2b}(\mathbf{p}) = ie^2\hbar \int d^4x d^4x' \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3 2p_0'} \frac{d^3\mathbf{q}}{(2\pi\hbar)^3 2q_0} \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}$$
$$\Big\{\theta(t'-t)\Phi^*_{\mathbf{p}'}(x) \overleftrightarrow{D}^\mu \bar{\Phi}_{\mathbf{q}}(x)\bar{\Phi}_{\mathbf{q}}(x') \overleftrightarrow{D}'_\mu \Phi_{\mathbf{p}}(x')e^{ik\cdot(x-x')}\Big\}. \tag{430}$$

From the integrand of the particle loop $\mathcal{F}_{2a}$, we have

$$\left(\phi^*_{\mathbf{p}'}(t)e^{-i\mathbf{p}'\cdot\mathbf{x}/\hbar} \overleftrightarrow{D}_\mu \phi_{\mathbf{q}}(t)e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\phi^*_{\mathbf{q}}(t')e^{-i\mathbf{q}\cdot\mathbf{x}'/\hbar} \overleftrightarrow{D}'^\mu \phi_{\mathbf{p}}(t')e^{i\mathbf{p}\cdot\mathbf{x}'/\hbar}\right) e^{-ik\cdot(x-x')}$$
$$= \phi^*_{\mathbf{p}'}(t)\phi_{\mathbf{p}}(t')\left[-\hbar^2 \overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{t'} + (\mathbf{p}+\mathbf{q}-2\mathbf{V}(t))\cdot(\mathbf{p}+\mathbf{q}-2\mathbf{V}(t'))\right]\phi_{\mathbf{q}}(t)\phi^*_{\mathbf{q}}(t')$$
$$\times \frac{1}{\hbar^2}e^{-i(\mathbf{p}'-\mathbf{q}-\mathbf{K})\cdot\mathbf{x}/\hbar}e^{i(\mathbf{p}-\mathbf{q}-\mathbf{K})\cdot\mathbf{x}'/\hbar}e^{-iK(t-t')/\hbar}, \tag{431}$$

where we have used the antisymmetry of $\overleftrightarrow{\partial}$ and $\mathbf{K} = \hbar\mathbf{k}$. Also we recall that for a spatial component j we have $D_j = \partial_j - iV^j/\hbar$. Integration over $\mathbf{x}$ and $\mathbf{x}'$ gives the delta functions $(2\pi\hbar)^6\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{K})\delta^3(\mathbf{p}'-\mathbf{p})$. Integrating out these

delta functions using the $\mathbf{p}'$ and $\mathbf{k}$ integrals[20] we obtain, with $K = |\mathbf{p} - \mathbf{q}|$,

$$\mathcal{F}_{2a}(\mathbf{p}) = \frac{ie^2}{2p_0} \int \frac{dtdt'd^3\mathbf{q}}{(2\pi\hbar)^3 2q_0 2K} \theta(t-t') e^{-iK(t-t')/\hbar}$$
$$\times\, \phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t') \left[-\hbar^2 \overleftrightarrow{\partial}_t \overleftrightarrow{\partial}_{t'} + (\mathbf{p}+\mathbf{q}-2\mathbf{V}(t))\cdot(\mathbf{p}+\mathbf{q}-2\mathbf{V}(t'))\right] \phi_{\mathbf{q}}(t)\phi^*_{\mathbf{q}}(t')\,, \tag{432}$$

where the time differentiations only apply to the ϕ terms on that line. Similarly, from the integrand of $\mathcal{F}_{2b}$, we have

$$\left(\phi^*_{\mathbf{p}'}(t)e^{-i\mathbf{p}'\cdot\mathbf{x}/\hbar} \overleftrightarrow{D}_\mu\, \bar{\phi}^*_{\mathbf{q}}(t)e^{-i\mathbf{q}\cdot\mathbf{x}/\hbar}\bar{\phi}_{\mathbf{q}}(t')e^{i\mathbf{q}\cdot\mathbf{x}'/\hbar} \overleftrightarrow{D}'^{\mu}\, \phi_{\mathbf{p}}(t')e^{i\mathbf{p}\cdot\mathbf{x}'/\hbar}\right) e^{-ik\cdot(x-x')}$$
$$= \phi^*_{\mathbf{p}'}(t)\phi_{\mathbf{p}}(t') \left[-\hbar^2 \overleftrightarrow{\partial}_t \overleftrightarrow{\partial}_{t'} + (\mathbf{p}-\mathbf{q}-2\mathbf{V}(t))\cdot(\mathbf{p}-\mathbf{q}-2\mathbf{V}(t'))\right] \bar{\phi}^*_{\mathbf{q}}(t)\bar{\phi}_{\mathbf{q}}(t')$$
$$\times\, \frac{1}{\hbar^2} e^{-i(\mathbf{p}'+\mathbf{q}+\mathbf{K})\cdot\mathbf{x}/\hbar} e^{i(\mathbf{p}+\mathbf{q}+\mathbf{K})\cdot\mathbf{x}'/\hbar} e^{iK(t-t')/\hbar}\,, \tag{433}$$

and consequently

$$\mathcal{F}_{2b}(\mathbf{p}) = \frac{ie^2}{2p_0} \int \frac{dtdt'd^3\mathbf{q}}{(2\pi\hbar)^3 2q_0 2K} \theta(t'-t) e^{iK(t-t')/\hbar}$$
$$\times\, \phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t') \left[-\hbar^2 \overleftrightarrow{\partial}_t \overleftrightarrow{\partial}_{t'} + (\mathbf{p}-\mathbf{q}-2\mathbf{V}(t))\cdot(\mathbf{p}-\mathbf{q}-2\mathbf{V}(t'))\right] \bar{\phi}^*_{\mathbf{q}}(t)\bar{\phi}_{\mathbf{q}}(t')\,. \tag{434}$$

Using the symmetry of the $\mathbf{q}$ integration, we can take $\mathbf{q} \to -\mathbf{q}$. Recalling that from the semiclassical expansion, $\bar{\phi}_{-\mathbf{q}} = \phi_{\mathbf{q}}$ we thus obtain

$$\mathcal{F}_{2b}(\mathbf{p}) = \frac{ie^2}{2p_0} \int \frac{dtdt'd^3\mathbf{q}}{(2\pi\hbar)^3 2q_0 2K} \theta(t'-t) e^{iK(t-t')/\hbar}$$
$$\times\, \phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t') \left[-\hbar^2 \overleftrightarrow{\partial}_t \overleftrightarrow{\partial}_{t'} + (\mathbf{p}+\mathbf{q}-2\mathbf{V}(t))\cdot(\mathbf{p}+\mathbf{q}-2\mathbf{V}(t'))\right] \phi^*_{\mathbf{q}}(t)\phi_{\mathbf{q}}(t')\,. \tag{435}$$

To briefly summarize the current situation, the second part of the forward scattering amplitude is given by the sum of the amplitudes representing the

[20] Recall that $d^3\mathbf{k}/k_0 = \hbar^{-2} d^3\mathbf{K}/K$.

particle and antiparticle loops and can be written

$$\mathcal{F}_2(\mathbf{p}) = \frac{ie^2}{2p_0}\int \frac{d^3\mathbf{q}}{2q_0(2\pi\hbar)^3}\frac{1}{2K}\int dt dt'$$
$$\Big\{\theta(t-t')\left[\phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t')\overleftrightarrow{\mathcal{D}}_1(t,t',\mathbf{p},\mathbf{q})\phi_{\mathbf{q}}(t)\phi^*_{\mathbf{q}}(t')\right]e^{-iK(t-t')/\hbar}$$
$$+\,\theta(t'-t)\left[\phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t')\overleftrightarrow{\mathcal{D}}_1(t,t',\mathbf{p},\mathbf{q})\phi^*_{\mathbf{q}}(t)\phi_{\mathbf{q}}(t')\right]e^{iK(t-t')/\hbar}\Big\}\,, \quad (436)$$

where

$$\overleftrightarrow{\mathcal{D}}_1(t,t',\mathbf{p},\mathbf{q}) \equiv -\hbar^2\,\overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{t'} + [\mathbf{p}+\mathbf{q}-2\mathbf{V}(t)]\cdot[\mathbf{p}+\mathbf{q}-2\mathbf{V}(t')]\,, \quad (437)$$

is the differential operator found in both loops.[21] In order to proceed further, it is convenient to define the new time variables $\bar{t}$ and η as follows:

- $t = \bar{t} - \dfrac{\hbar\eta}{2}$,
- $t' = \bar{t} + \dfrac{\hbar\eta}{2}$.

The Jacobian is straightforward: $dtdt' = \hbar\, d\bar{t}d\eta$. For the differential operator $\overleftrightarrow{\mathcal{D}}_1$, we note that

$$[\mathbf{p}+\mathbf{q}-\mathbf{V}(t)]\cdot[\mathbf{p}+\mathbf{q}-\mathbf{V}(t')] = [\mathbf{p}+\mathbf{q}-\mathbf{V}(\bar{t})]^2 + \mathcal{O}(\hbar^2)\,. \quad (438)$$

Including the Heaviside functions in our current considerations, we find that the amplitude $\mathcal{F}_2(\mathbf{p})$ can be rewritten as follows:

$$\mathcal{F}_2(\mathbf{p}) = \frac{ie^2}{2\hbar^2 p_0}\int \frac{d^3\mathbf{q}}{2q_0(2\pi)^3}\frac{1}{2K}\int d\bar{t}\,[G_-(\mathbf{p},\mathbf{q},\bar{t}) + G_+(\mathbf{p},\mathbf{q},\bar{t})]\,, \quad (439)$$

with

$$G_-(\mathbf{p},\mathbf{q},\bar{t}) = \int_{-\infty}^{0} d\eta\left[\phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t')\overleftrightarrow{\mathcal{D}}_2(\bar{t},\eta,\mathbf{p},\mathbf{q})\phi_{\mathbf{q}}(t)\phi^*_{\mathbf{q}}(t')\right]e^{iK\eta}\,, \quad (440)$$

$$G_+(\mathbf{p},\mathbf{q},\bar{t}) = \int_{0}^{\infty} d\eta\left[\phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t')\overleftrightarrow{\mathcal{D}}_2(\bar{t},\eta,\mathbf{p},\mathbf{q})\phi^*_{\mathbf{q}}(t)\phi_{\mathbf{q}}(t')\right]e^{-iK\eta}\,, \quad (441)$$

where

$$\overleftrightarrow{\mathcal{D}}_2(\bar{t},\eta,\mathbf{p},\mathbf{q}) \equiv -\frac{\hbar^2}{4}\,\overleftrightarrow{\partial}^2_{\bar{t}} - \overleftrightarrow{\partial}^2_{\eta} + [\mathbf{p}+\mathbf{q}-2\mathbf{V}(\bar{t})]^2 + \mathcal{O}(\hbar^2)\,. \quad (442)$$

[21] This was the main reason for the earlier manipulations of the $\mathbf{q}$ integration; to show that the operator can in fact be written the same in both loops.

The G_- term results from the particle loop contribution and the G_+ term from the antiparticle loop. The reader will notice that in the definitions for $G_\pm$ we have yet to convert the ϕ's to the new time variables. This task is the more complicated, involving the semiclassical expansion, and the evaluation of $G_\pm$ forms the bulk of our work in finding the forward scattering amplitude. Thus we have presented the above definitions first for aid of presentation. Apropos the mode function, the semiclassical expansion of the time-dependent factor $\phi_{\mathbf{q}}(t)$, can be written

$$\phi_{\mathbf{q}}(t) = \sqrt{\frac{q_0}{E_q(t)}}\varphi_{\mathbf{q}}(t) \exp\left[-\frac{i}{\hbar}\int_0^t E_q(\zeta)\, d\zeta\right] , \tag{443}$$

where the higher order $\hbar$ corrections are contained within $\varphi_{\mathbf{q}}(t)$ and thus we require $\varphi_{\mathbf{q}}(t) \to 1$ as $\hbar \to 0$. Substitution into the field equation (133) for $\phi(t)$ gives the $\hbar$ expansion of $\varphi_{\mathbf{q}}(t)$ as

$$\varphi_{\mathbf{q}}(t) = 1 + i\hbar\varphi_{\mathbf{q}}^{(1)}(t) + \mathcal{O}(\hbar^2) . \tag{444}$$

The explicit form of $\varphi_{\mathbf{q}}^{(1)}(t)$ will actually be unnecessary for our calculations, though it can easily be found. Note also that

$$\phi^*_{\mathbf{q}}(t)\phi_{\mathbf{q}}(t') = \frac{q_0\varphi^*_{\mathbf{q}}(t)\varphi_{\mathbf{q}}(t')}{\sqrt{E_q(t)E_q(t')}} \exp\left[-i\int_{-\eta/2}^{\eta/2} E_q(\bar{t}+\hbar\zeta)\, d\zeta\right] . \tag{445}$$

We note the following: Converting to $(\bar{t}, \eta)$ variables we have $\varphi^*_{\mathbf{q}}(t)\varphi_{\mathbf{q}}(t') = 1 + \mathcal{O}(\hbar^2)$ and $\sqrt{E_q(t)E_q(t')} = E_q(\bar{t}) + \mathcal{O}(\hbar^2)$. We also note the lack of an order $\hbar$ term in (442) because

$$[\mathbf{p}+\mathbf{q}-\mathbf{V}(t)]\cdot[\mathbf{p}+\mathbf{q}-\mathbf{V}(t')] = [\mathbf{p}+\mathbf{q}-\mathbf{V}(\bar{t})]^2 + \mathcal{O}(\hbar^2) . \tag{446}$$

With these relations in mind, it can readily be shown that the functions $G_\pm(\mathbf{p},\mathbf{q},t)$ are of the form

$$\begin{aligned} G_\pm(\mathbf{p},\mathbf{q},t) = \pm\int_0^{\pm\infty} d\eta \left[f_\pm(\mathbf{p},\mathbf{q},t) + \mathcal{O}(\hbar^2)\right] \\ \times \exp\left\{\mp i\int_{-\eta/2}^{+\eta/2} d\zeta \left[\pm E_p(t+\hbar\zeta) + E_q(t+\hbar\zeta) + K\right]\right\} , \end{aligned} \tag{447}$$

where after performing the appropriate differentiations, the function $f_{\pm}(\mathbf{p}, \mathbf{q}, t)$ can be found as

$$f_{\pm}(\mathbf{p}, \mathbf{q}, t) = \left\{ - \left[E_p(t) \mp E_q(t) \right]^2 + \left[\mathbf{p} + \mathbf{q} - 2\mathbf{V}(t) \right]^2 \right\} |\phi_{\mathbf{p}}(t)|^2 |\phi_{\mathbf{q}}(t)|^2 . \quad (448)$$

The points noted above conspire to produce the important fact that there are no terms of order $\hbar$ in the pre-factor of (447), inside the first square brackets. Having now removed the last trace of the original time variables, we may clean up the notation by changing $\bar{t} \to t$ and consider the evaluation of the above $G_{\pm}$ integrals.

Let us first consider the integral $G_{+}(\mathbf{p}, \mathbf{q}, t)$. We change the integration variable from η to β defined by the following relation:

$$[E_p(t) + E_q(t) + K]\beta \equiv \int_{-\eta/2}^{\eta/2} [E_p(t + \hbar\zeta) + E_q(t + \hbar\zeta) + K] \, d\zeta . \quad (449)$$

Expanding the integrand and integrating the right hand side, we obtain

$$\begin{aligned} [E_p(t) + E_q(t) + K] \, \beta = [E_p(t) + E_q(t) + K] \, \eta + \hbar^2 \frac{\eta^3}{3.2^3} \left[\ddot{E}_p(t) + \ddot{E}_q(t) \right] \\ + \mathcal{O}(\hbar^4 \eta^5) . \end{aligned} \quad (450)$$

From the above it is evident that $\eta = \beta + \mathcal{O}(\hbar^2)$. Hence we solve this equation for η as a function of β for small $\hbar$ and find

$$\eta = \left[1 - \frac{1}{24} \frac{\ddot{E}_p(t) + \ddot{E}_q(t)}{E_p(t) + E_q(t) + K} \hbar^2 \beta^2 + \mathcal{O}(\hbar^4 \beta^4) \right] \beta , \quad (451)$$

and

$$d\eta = \left[1 - \frac{1}{8} \frac{\ddot{E}_p(t) + \ddot{E}_q(t)}{E_p(t) + E_q(t) + K} \hbar^2 \beta^2 + \mathcal{O}(\hbar^4 \beta^4) \right] d\beta . \quad (452)$$

We label the Jacobian $\hbar$ expansion contained in the square brackets by $J(\mathbf{p}, \mathbf{q}, t, \hbar\beta)$. The integral we are concerned with is then

$$G_{+}(\mathbf{p}, \mathbf{q}, t) = \int_0^{\infty} d\beta \left[f_{+}(\mathbf{p}, \mathbf{q}, t) + \mathcal{O}(\hbar^2) \right] \exp \left\{ -i \left[E_p(t) + E_q(t) + K \right] \beta \right\} . \quad (453)$$

The integration can be completed if we introduce a convergence factor by replacing K with $K - i\epsilon$. Accordingly, we obtain

$$G_+(\mathbf{p}, \mathbf{q}, t) = -\frac{if_+(\mathbf{p}, \mathbf{q}, t)}{E_q(t) + E_p(t) + K} + \mathcal{O}(\hbar^2)\,. \tag{454}$$

The corresponding contribution to the forward-scattering amplitude can be seen with reference to (439) as[22]

$$\mathcal{F}_{2+}(\mathbf{p}) = \frac{e^2}{2\hbar^2 p_0} \int dt \int \frac{d^3\mathbf{q}}{2q_0(2\pi)^3} \frac{1}{2K} \frac{f_+(\mathbf{p}, \mathbf{q}, t)}{E_p(t) + E_q(t) + K} + \mathcal{O}(\hbar^0)\,, \tag{455}$$

The amplitude $\mathcal{F}_{2+}(\mathbf{p})$ can readily be seen to be ultra-violet divergent. This will not however be a cause of difficulty, as we fully expect the results to be divergent.

Next we analyse the contribution from $G_-(\mathbf{p}, \mathbf{q}, t)$, with which we find ourselves additional difficulties. One cannot proceed as above because of the infrared divergence in the $\mathbf{q}$-integration as we shall see shortly. We start as in the previous case and define the variable $\tilde{\beta}$ in analogy with the variable β in (449) as follows

$$[-E_p(t) + E_q(t) + K]\tilde{\beta} \equiv \int_{-\eta/2}^{\eta/2} [-E_p(t + \hbar\zeta) + E_q(t + \hbar\zeta) + K]\; d\zeta\,. \tag{456}$$

With foresight knowledge of the new divergence as $K \to 0$ [23] we should check the validity of using the variable $\tilde{\beta}$ in such circumstances. For small $K = \|\mathbf{p} - \mathbf{q}\|$, we have

$$-E_p(t) + E_q(t) + K \approx K - \mathbf{v}(t) \cdot \mathbf{K}\,, \tag{457}$$

where $\mathbf{v}(t) = [\mathbf{p} - \mathbf{V}(t)]/E_p(t)$ is the velocity of the classical particle with final momentum $\mathbf{p}$. [24] Hence, in the limit $K \to 0$ one finds

$$\tilde{\beta} = \frac{1}{1 - \mathbf{v}(t) \cdot \mathbf{n}} \int_{-\eta/2}^{\eta/2} [1 - \mathbf{v}(t + \hbar\zeta) \cdot \mathbf{n}]\, d\zeta\,, \tag{458}$$

[22] Recall the change of notation $\bar{t} \to t$ since that reference.

[23] Recall $d^3\mathbf{q} = d^3\mathbf{K}$.

[24] See (339) for this result.

where $\mathbf{n} \equiv \mathbf{K}/K$. Thus, if we write $d\eta = \tilde{J}(\mathbf{p}, \mathbf{q}, t, \hbar\tilde{\beta})d\tilde{\beta}$, then the function $\tilde{J}(\mathbf{p}, \mathbf{q}, t, \hbar\tilde{\beta})$ is finite as $K \to 0$ and we can now safely use the $\tilde{\beta}$ definition (456). The expression corresponding to (453) can be given in the following form:

$$
\begin{aligned}
G_-(\mathbf{p}, \mathbf{q}, t) &= \int_{-\infty}^{0} d\tilde{\beta} \left[f_-(\mathbf{p}, \mathbf{q}, t) + \sum_{n,d} \hbar^n \tilde{\beta}^d f_{nd-}(\mathbf{p}, \mathbf{q}, t) \right] \\
&\qquad \times \exp\left\{ i \left[-E_p(t) + E_q(t) + K \right] \tilde{\beta} \right\} \\
&= \frac{-i f_-(\mathbf{p}, \mathbf{q}, t)}{-E_p(t) + E_q(t) + K} + \sum_{n,d} \frac{(-i)^d d! \hbar^n f_{nd-}(\mathbf{p}, \mathbf{q}, t)}{\left[-E_p(t) + E_q(t) + K \right]^{d+1}}, \qquad (459)
\end{aligned}
$$

with $n \geq 2$ and $n \geq d \geq 0$. The higher order pre-factors $f_{nd-}(\mathbf{p}, \mathbf{q}, t)$, which are finite as $K \to 0$, can not be removed in the $\hbar \to 0$ limit as for G_+ due to the infrared divergence. This can be seen clearly if we substitute (459) into the amplitude expression (439):

$$
\begin{aligned}
\mathcal{F}_{2-}(\mathbf{p}) &= \frac{ie^2}{2\hbar^2 p_0} \int \frac{d^3\mathbf{q}}{(2\pi)^3 2q_0} \frac{1}{2K} \int dt \\
&\left[\frac{-i f_-(\mathbf{p}, \mathbf{q}, t)}{-E_p(t) + E_q(t) + K} + \sum_{n,d} \frac{(-i)^d d! \hbar^n f_{nd-}(\mathbf{p}, \mathbf{q}, t)}{\left[-E_p(t) + E_q(t) + K \right]^{d+1}} \right]. \qquad (460)
\end{aligned}
$$

Here we see that the terms with $d \geq 1$ are infrared divergent in the $\mathbf{q}$-integration because of the limit $\lim_{K\to 0}[-E_p(t) + E_q(t) + K] \to 0$ as can be seen from (457).

We can approach this difficulty by separating the infrared divergent section of the integral via the addition of a cut-off in the $\mathbf{q}$ (or equivalently $\mathbf{K}$) integral. We may then consider the situation above the cut-off and return to the problematic sub-cut-off area later. Let us thus cut-off the integral by requiring

$$
K \geq K_0 = \hbar^\alpha \lambda, \qquad (461)
$$

with λ a positive constant and where $\frac{3}{4} < \alpha < 1$. The reasoning for choosing these precise limits on the choice of α will become apparent in later stages of the calculations. Above the cut-off K_0, we find that the contributions of the

terms of $\mathcal{F}_{2-}$ to the $\mathbf{q}$ integration have the small-K behaviour

$$\begin{cases} \hbar^n K_0^{1-d} = \hbar^{n+(1-d)\alpha}\lambda^{1-d} & \text{if } d \geq 2 \\ \hbar^n \log(\hbar^\alpha \lambda) & \text{if } d = 1 \end{cases} . \tag{462}$$

Since $1-\alpha > 0$, $n \geq 2$ and $n \geq d$, from their appropriate definitions, we have

$$n + (1-d)\alpha \geq 2 - \alpha\,, \tag{463}$$

for the $\hbar$ power in the $d \geq 2$ case above.[25] We thus see that in the $\hbar \to 0$ limit, the higher order f_{nd-} terms will not contribute above the cut-off.[26] In this arena we are thus left only with the leading order term containing f_-, in a situation analogous to $\mathcal{F}_{2+}$. We give this leading order contribution, over the full range both above and below the cut-off, the label $\mathcal{F}^0_{2-}$,

$$\mathcal{F}^0_{2-} = \frac{e^2}{2\hbar^2 p_0} \int dt \int \frac{d^3\mathbf{q}}{2q_0(2\pi)^3} \frac{1}{2K} \frac{f_-(\mathbf{p},\mathbf{q},t)}{(-E_p(t) + E_q(t) + K)}\,. \tag{464}$$

For $\mathcal{F}_{2+}$, we effectively have $\mathcal{F}_{2+} = \mathcal{F}^0_{2+}$ to order $\hbar^{-1}$ (which is the highest order we require).

At this point we pause to take stock of the various contributions to the forward scattering. Firstly we have the leading order terms. We combine $\mathcal{F}_1$, $\mathcal{F}^0_{2+}$ and $\mathcal{F}^0_{2-}$ and let

$$\mathcal{F}^0 = \mathcal{F}_1 + \mathcal{F}^0_{2+} + \mathcal{F}^0_{2-}\,. \tag{465}$$

What remains is the contribution of the higher order terms from $\mathcal{F}_{2-}$, from below the cut-off. We shall attack this in a round-about way. Labelling the full $\mathcal{F}_{2-}$ term below the cut-off by $\mathcal{F}^<_{2-}$, and the leading order term in the same range by $\mathcal{F}^{<,0}_{2-}$, the desired contribution can be calculated as

$$\mathcal{F}^{<,\mathrm{ho}} = \mathcal{F}^<_{2-} - \mathcal{F}^{<,0}_{2-}\,. \tag{466}$$

[25] $n + (1-d)\alpha \geq n + (1-n)\alpha = n(1-\alpha) + \alpha \geq 2(1-\alpha) - \alpha = 2 - \alpha\,.$

[26] This explains our choice of the upper limit on α.

Thus the forward scattering amplitude[27] is given by

$$\mathcal{F}(\mathbf{p}) = \mathcal{F}^0(\mathbf{p}) + \mathcal{F}^{<,\mathrm{ho}}(\mathbf{p})\,. \tag{467}$$

With this aside complete, we now return to finish the calculation of these two terms.

Firstly we turn to the higher order contributions $\mathcal{F}^{<,\mathrm{ho}}$ which we approach as described above. For these terms, we are interested in their behaviour at small-k. For the full-term $\mathcal{F}^{<}_{2-}$, we consequently back-track somewhat to the expression in (436), the appropriate part of which gives

$$\begin{aligned}\mathcal{F}^{<}_{2}(\mathbf{p}) = \frac{ie^2}{2p_0}\int_{<}\frac{d^3\mathbf{q}}{2q_0(2\pi\hbar)^3}\frac{1}{2K}\int dtdt'\theta(t-t')\\ \times\left[\phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t')\overset{\leftrightarrow}{\mathcal{D}}_1(t,t',\mathbf{p},\mathbf{q})\phi_{\mathbf{q}}(t)\phi^*_{\mathbf{q}}(t')\right]e^{-iK(t-t')/\hbar}\,.\end{aligned} \tag{468}$$

The subscript $<$ on the $\mathbf{q}$ integral indicates that we integrate below the cut-off. We change the integration variable, firstly to the photon momentum $\mathbf{K} = \mathbf{p}-\mathbf{q}$ and then to the wave number $\mathbf{k} = \mathbf{K}/\hbar$. Ergo $d^3\mathbf{q}/(\hbar^3 K) = d^3\mathbf{k}/(\hbar k)$ and write

$$\begin{aligned}\mathcal{F}^{<}_{2}(\mathbf{p}) = \frac{ie^2}{2\hbar p_0}\int dtdt'\int_{k\le\hbar^{\alpha-1}\lambda}\frac{d^3\mathbf{k}}{2q_0(2\pi)^3}\frac{1}{2k}\theta(t-t')\\ \times\left[\phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t')\overset{\leftrightarrow}{\mathcal{D}}_1(t,t',\mathbf{p},\mathbf{q})\phi_{\mathbf{q}}(t)\phi^*_{\mathbf{q}}(t')\right]e^{-ik(t-t')}\,.\end{aligned} \tag{469}$$

In the $\hbar\to 0$ limit we note that the upper limit $\hbar^{\alpha-1}\lambda$ of integration for k becomes infinite. Now we have $\mathbf{q} = \mathbf{p}-\hbar\mathbf{k}$. Hence, we have $\mathbf{q}\to\mathbf{p}$ for all $\mathbf{k}$ as $\hbar\to 0$ because $\hbar\cdot\hbar^{\alpha-1}\lambda\to 0$. Using these limits, and with reference to our previous calculations for $\mathcal{F}_{2-}$, the exponential factor can take the form

$$\begin{aligned}&\exp\left\{i\int_t^{t'} d\zeta\left[K + E_q(\zeta) - E_p(\zeta)\right]/\hbar\right\}\\ &\qquad= \exp\left\{i\int_t^{t'}\left[k - \frac{\partial E_p(\zeta)}{\partial\mathbf{p}}\cdot\mathbf{k} + \frac{1}{2}\frac{\partial^2 E_p(\zeta)}{\partial p^i\partial p^j}\hbar k^i k^j + \cdots\right]d\zeta\right\}\,.\end{aligned} \tag{470}$$

In order to truncate the series in the exponent at the second term for all $\mathbf{k}$ in the integration range, one would require $\hbar(\hbar^{\alpha-1})^2\to 0$ as $\hbar\to 0$ i.e. $\alpha > \frac{1}{2}$.

[27]non-renormalised so far,

This is naturally satisfied due to the earlier choice of $\alpha > \frac{3}{4}$. Thus for the $\mathbf{q} \to \mathbf{p}$ limit we write

$$
\begin{aligned}
\mathcal{F}_2^<(\mathbf{p}) = &\frac{ie^2}{\hbar 2p_0} \int dt dt' \int_{k \leq \hbar^{\alpha-1}\lambda} \frac{d^3\mathbf{k}}{2p_0(2\pi)^3} \frac{1}{2k} \theta(t-t') \\
&\times \left[-2E_p(t) 2E_p(t') + 4\left(\mathbf{p} - \mathbf{V}(t)\right) \cdot \left(\mathbf{p} - \mathbf{V}(t')\right)\right] |\phi_{\mathbf{p}}(t)|^2 |\phi_{\mathbf{p}}(t')|^2 \\
&\exp\left[ik(t'-t) - i\int_t^{t'} \frac{\mathbf{p} - \mathbf{V}(\zeta)}{E_p(\zeta)} \cdot \mathbf{k} d\zeta\right] \left(1 + \mathcal{O}(\hbar k^2)\right) \\
= &\frac{ie^2}{\hbar} \int dt dt' \int_{k \leq \hbar^{\alpha-1}\lambda} \frac{d^3\mathbf{k}}{(2\pi)^3 2k} \theta(t-t') \\
&\times \left[-1 + \frac{(\mathbf{p} - \mathbf{V}(t)) \cdot (\mathbf{p} - \mathbf{V}(t'))}{E_p(t) E_p(t')}\right] \\
&\exp\left[ik(t'-t) - i\int_t^{t'} \frac{\mathbf{p} - \mathbf{V}(\zeta)}{E_p(\zeta)} \cdot \mathbf{k} d\zeta\right] + \mathcal{O}(\hbar^{4\alpha-4}) \,,
\end{aligned} \tag{471}
$$

where we have used $|\phi_{\mathbf{p}}(t)|^2 = p_0/E_p(t) + \mathcal{O}(\hbar^2)$ and recall from (336) that the local momentum is $md\mathbf{x}/d\tau = \mathbf{p} - \mathbf{V}$. The extra $\mathcal{O}(\hbar^{4\alpha-4})$ is the result of the combination of $\hbar k^2$ from the higher order dependence with the $k/\hbar$ dependence multiplying the entire integrand. These non-leading terms do not contribute to the position shift provided the overall order is greater than $\hbar^{-1}$, which is the case since $4\alpha - 4 > -1$ because of our earlier requirement that $\alpha > \frac{3}{4}$. We consequently drop this contribution from now on. Recalling that $[\mathbf{p} - \mathbf{V}(t)]/E_p(t)$ is the velocity of the corresponding classical particle, $d\mathbf{x}/dt$, we obtain at leading order

$$
\begin{aligned}
\mathcal{F}_2^<(\mathbf{p}) &= \frac{ie^2}{\hbar} \int_{k \leq \hbar^{\alpha-1}\lambda} \frac{d^3\mathbf{k}}{(2\pi)^3 2k} dt dt' \theta(t-t') \left[-\frac{dx^\mu}{dt} \frac{dx_\mu}{dt'}\right] e^{i(k(t'-t) - \mathbf{k}\cdot(\mathbf{x}(t) - \mathbf{x}(t')))} \\
&= -\frac{ie^2}{\hbar} \int_{k \leq \hbar^{\alpha-1}\lambda} \frac{d^3\mathbf{k}}{(2\pi)^3 2k} \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} d\xi' \theta(\xi - \xi') \frac{dx^\mu}{d\xi} \frac{dx_\mu}{d\xi'} e^{ik(\xi' - \xi)} \,,
\end{aligned} \tag{472}
$$

where in analogy with the emission amplitude, we have defined $\xi \equiv t_1 - \mathbf{n} \cdot \mathbf{x}(t_1)$ and $\xi' \equiv t_2 - \mathbf{n} \cdot \mathbf{x}(t_2)$ with $\mathbf{n} \equiv \mathbf{k}/k$.

The Heaviside function can be rewritten in the form

$$
\theta(\xi - \xi') = \frac{1}{2} + \frac{1}{2}\epsilon(\xi - \xi') \,, \tag{473}
$$

where[28]

$$\epsilon(\xi - \xi') \equiv \begin{cases} 1 & \text{if } \xi > \xi' \\ -1 & \text{if } \xi < \xi' \end{cases} . \tag{474}$$

The use of (473) in place of the step function in $\mathcal{F}_2^<$ in (472) has the advantage that it splits the real and imaginary parts of the expression. The reader will recall that only the real part of the forward scattering amplitude affects the position shift. Also worthy of recall is the equivalence demonstrated between the emission probability and the imaginary part of $\mathcal{F}(\mathbf{p})$ as given by (314). Using the above, we find twice the imaginary part as

$$2\Im\,\mathcal{F}^<(\mathbf{p}) = \frac{e^2}{\hbar}\int_{k\le\hbar^{\alpha-1}\lambda}\frac{d^3\mathbf{k}}{(2\pi)^3 2k}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}d\xi'\frac{dx^\mu}{d\xi}\frac{dx_\mu}{d\xi'}e^{ik(\xi'-\xi)}\,. \tag{475}$$

In the limit $\hbar^{\alpha-1}\lambda \to \infty$ this expression coincides with the emission probability using our expression for the emission amplitude, as required by unitarity.[29] The real part can similarly be written

$$\begin{aligned}&\Re\,\mathcal{F}_2^<(\mathbf{p})\\ &= \frac{ie^2}{2\hbar}\int_{k\le\hbar^{\alpha-1}\lambda}\frac{kdkd\Omega}{2(2\pi)^3}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}d\xi'\epsilon(\xi'-\xi)\frac{dx^\mu}{d\xi}\chi(\xi)\frac{dx_\mu}{d\xi'}\chi(\xi')e^{ik(\xi'-\xi)}\,.\end{aligned} \tag{476}$$

Here we have again introduced the cut-off function $\chi(\xi)$ as defined in section 3.3. In addition, we have used the antisymmetry of the sign function $-\epsilon(\xi - \xi') = \epsilon(\xi'-\xi)$ and expanded the $d^3\mathbf{k}$ in spherical polar coordinates. Integration by parts with respect to ξ' gives

$$\begin{aligned}\Re\,\mathcal{F}_2^<(\mathbf{p}) = -\frac{ie^2}{2\hbar}\int_{k\le\hbar^{\alpha-1}\lambda}\frac{kdkd\Omega}{2(2\pi)^3}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}d\xi'&\\ \frac{d}{d\xi'}\left[\epsilon(\xi'-\xi)\frac{dx_\mu}{d\xi'}\chi(\xi')\right]\frac{dx^\mu}{d\xi}\chi(\xi)\frac{e^{ik(\xi'-\xi)}}{ik}\,.&\end{aligned} \tag{477}$$

[28]The function $\epsilon(x)$ is the sign function and sometimes written $\mathrm{sgn}(x)$.

[29]This demonstrates the semiclassical approximation for the emission probability and thus validates the previous physically reasonable assumption that a typical photon energy emitted has energy of order $\hbar$.

Alternatively, we can integrate by parts with respect to ξ to obtain a similar result. Adding the two expressions and dividing by two we obtain the symmetrized version

$$\Re\,\mathcal{F}_2^<(\mathbf{p}) = -\frac{e^2}{4\hbar}\int_{k\leq\hbar^{\alpha-1}\lambda}\frac{dkd\Omega}{2(2\pi)^3}\int_{-\infty}^{+\infty}d\xi\int_{-\infty}^{+\infty}d\xi'\left\{4\delta(\xi'-\xi)\frac{dx^\mu}{d\xi'}\frac{dx_\mu}{d\xi}\right.$$
$$\left.+\epsilon(\xi'-\xi)\left[\left(\frac{d}{d\xi'}-\frac{d}{d\xi}\right)\frac{dx^\mu}{d\xi'}\chi(\xi')\frac{dx_\mu}{d\xi}\chi(\xi)\right]\right\}e^{ik(\xi'-\xi)}\,, \quad (478)$$

where we have used the result $d\epsilon(x)/dx = 2\delta(x)$ and we have taken the limit $\chi(\xi)\to 1$ for the first term. Consider for a moment the second term of (478): Due to the $\xi \leftrightarrow \xi'$ symmetry of the factors inside the curly brackets, in the $\hbar\to 0$ limit we can extend the integration range of k from $(0,\infty)$ to $(-\infty,+\infty)$ and divide by two once more. Consequently the k integration produces the delta function $\delta(\xi'-\xi)$. The second term of (478) is zero when $\xi=\xi'$ and consequently we can say that the contribution from this term to $\Re\mathcal{F}_2^<$ is of order higher than $\hbar^{-1}$. The first term thus remains which we can rewrite as

$$\begin{aligned}\Re\,\mathcal{F}_2^<(\mathbf{p}) &= -\frac{e^2}{2(2\pi)^3\hbar}\int_{k\leq\hbar^{\alpha-1}\lambda}dkd\Omega d\xi\frac{dx^\mu}{d\xi}\frac{dx_\mu}{d\xi}\\ &= -\frac{e^2}{16\pi^3\hbar}\int_{k\leq\hbar^{\alpha-1}\lambda}dkd\Omega dt\frac{dx^\mu}{dt}\frac{dx_\mu}{dt}\frac{dt}{d\xi}\\ &= -\frac{e^2\lambda}{16\pi^3\hbar^{2-\alpha}}\int_{-\infty}^{+\infty}dt\int d\Omega\frac{1-\mathbf{v}^2}{1-\mathbf{n}\cdot\mathbf{v}}\,,\end{aligned} \quad (479)$$

to order $\hbar^{-1}$. We have integrated over k and noted that $\dot\xi = 1-\mathbf{n}\cdot\mathbf{v}$ and $(dx^\mu/dt)(dx_\mu/dt) = 1-\mathbf{v}^2$.

We now turn our attention to the leading order term of $\mathcal{F}_{2-}$ below the cut-off, which by expanding f_- in (464) is given by

$$\mathcal{F}_2^{<,0} = \frac{e^2}{2\hbar^2 p_0}\int dt\int_{K\leq\hbar^\alpha\lambda}\frac{d^3\mathbf{q}}{2q_0(2\pi)^3}\frac{1}{2K}$$
$$\frac{-\left(E_p((t)+E_q(t)\right)^2+\left(\mathbf{p}+\mathbf{q}-2\mathbf{V}(t)\right)^2}{(-E_p(t)+E_q(t)+K)}|\phi_{\mathbf{p}}(t)|^2|\phi_{\mathbf{q}}(t)|^2\,. \quad (480)$$

Using the small-K approximation (457) and noting the following equations

$$|\phi_{\mathbf{p}}(t)|^2 = \frac{p_0}{E_p(t)} + \mathcal{O}(\hbar^2)\,, \tag{481}$$

$$\frac{\mathbf{p}-\mathbf{V}(t)}{E_p(t)} = \mathbf{v}\,, \tag{482}$$

we find

$$\begin{aligned}
\mathcal{F}_2^{<,0} &= \frac{e^2}{2(2\pi)^3\hbar^2}\int dt \int_{K\le\hbar^\alpha\lambda} \frac{d^3\mathbf{q}}{K}\frac{1}{4E_p^2(t)}\frac{-4E_p^2+4\left(\mathbf{p}-\mathbf{V}(t)\right)^2}{K\left(1-\mathbf{n}\cdot\mathbf{v}\right)} \\
&= -\frac{e^2}{16\pi^3\hbar^2}\int dt \int_{K\le\hbar^\alpha\lambda} dK d\Omega \frac{1-\mathbf{v}^2}{(1-\mathbf{n}\cdot\mathbf{v})} \\
&= -\frac{e^2\lambda}{16\pi^3\hbar^{2-\alpha}}\int_{-\infty}^{+\infty} dt \int d\Omega \frac{1-\mathbf{v}^2}{1-\mathbf{n}\cdot\mathbf{v}}\,.
\end{aligned} \tag{483}$$

We recognize the same expression arrived at in this limit for $\mathcal{F}_2^<$ in (479).

From the above results we thus conclude that $\mathcal{F}_2^{<,0}$ is equal to the leading term of $\mathcal{F}_2^<$. Hence $\mathcal{F}^{<,\mathrm{ho}} = \mathcal{F}_2^< - \mathcal{F}_2^{<,0}$ is of order $\hbar^{-1}$, but is purely imaginary at this order. Due to the fact that only the real part of the forward scattering affects the position shift, the only remaining contributions are those grouped under $\mathcal{F}^0$ and it is these terms to which we now draw our attention.

The leading order part of the forward scattering amplitude was earlier defined by $\mathcal{F}^0 = \mathcal{F}_1 + \mathcal{F}_{2+}^0 + \mathcal{F}_{2-}^0$ where we have so far found that

$$\mathcal{F}_1(\mathbf{p}) = -\frac{3e^2}{2\hbar^2 p_0}\int dt |\phi_{\mathbf{p}}(t)|^2 \int \frac{d^3\mathbf{K}}{(2\pi)^3 2K} \tag{417}$$

$$\mathcal{F}_{2+}(\mathbf{p}) = \frac{e^2}{2\hbar^2 p_0}\int dt \int \frac{d^3\mathbf{q}}{2q_0(2\pi)^3}\frac{1}{2K}\frac{f_+(\mathbf{p},\mathbf{q},t)}{E_p(t)+E_q(t)+K} + \mathcal{O}(\hbar^0) \tag{455}$$

$$\mathcal{F}_{2-}^0(\mathbf{p}) = \frac{e^2}{2\hbar^2 p_0}\int dt \int \frac{d^3\mathbf{q}}{2q_0(2\pi)^3}\frac{1}{2K}\frac{f_-(\mathbf{p},\mathbf{q},t)}{(-E_p(t)+E_q(t)+K)}\,. \tag{464}$$

Substituting in the expressions for $f_\pm$ the $\mathcal{F}_{2\pm}$ terms are[30]

$$
\begin{aligned}
&\mathcal{F}_{2\pm} \\
&= \frac{e^2}{2\hbar^2 p_0}\int dt \int \frac{d^3\mathbf{q}}{2q_0(2\pi)^3}\frac{1}{2K} \\
&\quad \frac{\left[-\left(E_p \mp E_q\right)^2 + \left(\mathbf{p}+\mathbf{q}-2\mathbf{V}\right)^2\right]}{\left(\pm E_p + E_q + K\right)}|\phi_{\mathbf{p}}(t)|^2|\phi_{\mathbf{q}}(t)|^2 \\
&= \frac{e^2}{2\hbar^2 p_0}\int dt|\phi_{\mathbf{p}}(t)|^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3 2E_q(t)}\frac{1}{2K}\frac{\left[-\left(E_p \mp E_q\right)^2 + \left(\mathbf{p}+\mathbf{q}-2\mathbf{V}\right)^2\right]}{\left(\pm E_p + E_q + K\right)}\,.
\end{aligned}
\tag{484}
$$

We slightly modify the expression for $\mathcal{F}_1$, using the variable of integration $\mathbf{q} = \mathbf{p} - \mathbf{K}$ defined and used in the $\mathcal{F}_2$'s, to bring it in line with the others in form.

$$
\mathcal{F}_1(\mathbf{p}) = \frac{e^2}{2\hbar^2 p_0}\int dt|\phi_{\mathbf{p}}(t)|^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3 2E_q(t)}\frac{1}{2K}\left[-6E_q(t)\right]\,. \tag{485}
$$

Combining the terms of $\mathcal{F}^0(\mathbf{p})$, we therefore write

$$
\begin{aligned}
&\mathcal{F}^0(\mathbf{p}) \\
&= \frac{e^2}{2\hbar^2 p_0}\int dt|\phi_{\mathbf{p}}(t)|^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3 2E_q}\frac{1}{2K}\Bigg[-6E_q \\
&\quad + \frac{\left[-\left(E_p + E_q\right)^2 + \left(\mathbf{p}+\mathbf{q}-2\mathbf{V}\right)^2\right]}{\left(-E_p + E_q + K\right)} + \frac{\left[-\left(E_p - E_q\right)^2 + \left(\mathbf{p}+\mathbf{q}-2\mathbf{V}\right)^2\right]}{\left(E_p + E_q + K\right)}\Bigg]\,.
\end{aligned}
\tag{486}
$$

We define the momenta $\tilde{\mathbf{p}} = \mathbf{p} - \mathbf{V}(t)$ and $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{V}(t)$, so that we have $E_p = \sqrt{\tilde{\mathbf{p}}^2 + m^2}$ and similarly for E_q. After these substitutions, we are free to change the integration variable from $\mathbf{q}$ to $\tilde{\mathbf{q}}$. This gives

$$
\begin{aligned}
\mathcal{F}^0(\mathbf{p}) = -&\frac{e^2}{2\hbar^2 p_0}\int dt|\phi_{\mathbf{p}}(t)|^2 \int \frac{d^3\tilde{\mathbf{q}}}{(2\pi)^3 2E_q}\frac{1}{2K} \\
&\left[6E_q + \frac{\left[\left(E_p + E_q\right)^2 - \left(\tilde{\mathbf{p}}+\tilde{\mathbf{q}}\right)^2\right]}{\left(-E_p + E_q + K\right)} + \frac{\left[\left(E_p - E_q\right)^2 - \left(\tilde{\mathbf{p}}+\tilde{\mathbf{q}}\right)^2\right]}{\left(E_p + E_q + K\right)}\right]\,.
\end{aligned}
\tag{487}
$$

[30]We have removed the (t) from the E's and $\mathbf{V}$'s for ease of presentation as there is no risk of confusion here.

This term is then our remaining contribution to the forward scattering amplitude. It is real and of order $\hbar^{-2}$, thus would contribute at order $\hbar^{-1}$ to the position shift. It is also divergent. We shall now show that this divergent contribution is exactly canceled by the contribution from the divergent mass counterterm when we renormalise the mass.

4.3. Renormalisation. We achieve renormalisation of the mass by the counterterm addition to the Lagrangian

$$\delta\mathcal{L} = \frac{\delta m^2}{\hbar^2}\varphi^\dagger\varphi\,. \tag{488}$$

This in turn provides an additional contribution to the interaction Hamiltonian that is included in the forward scattering, viz

$$\delta\mathcal{H}_I = -\frac{\delta m^2}{\hbar^2}\varphi^\dagger\varphi\,. \tag{489}$$

This term contributes at first order in $\mathcal{H}_I$[31] as in (411), thus

$$\begin{aligned}\delta\mathcal{F}(\mathbf{p}) &= \frac{1}{\hbar}\int\frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3}d^4x\frac{\delta m^2}{\hbar^2}\langle 0|A(\mathbf{p}'){:}\varphi^\dagger(x)\varphi(x){:}A^\dagger(\mathbf{p})\\ &= \frac{1}{\hbar}\int\frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3}d^4x\delta m^2\Phi^*_{\mathbf{p}'}(x)\Phi_{\mathbf{p}}(x)\\ &= \frac{1}{2\hbar p_0}\int dt|\phi_{\mathbf{p}}(t)|^2\delta m^2\,,\end{aligned} \tag{490}$$

where we used (415) for the inner product. To compute the counterterm we first find the self-energy $\Sigma(p)$. Using the Feynman rules for the standard covariant perturbation theory of scalar electrodynamics we obtain

$$\Sigma(p) = \frac{e^2}{\hbar}\int\frac{d^4q}{(2\pi)^4 i}\left\{\frac{(p+q)^2}{[q^2-m^2+i\epsilon]\,[(p-q)^2+i\epsilon]} - \frac{4}{[(p-q)^2+i\epsilon]}\right\}, \tag{491}$$

where we use $q = p - K$. The convergence factors $i\epsilon$ are added, along with $\epsilon > 0$ and the usual assumption that the limit $\epsilon \to 0$ is to be taken at the end.

[31] δm^2 is of order e^2 as will be seen shortly.

We integrate over the q_0 component in order to compare with our previous results. In the denominators we have the terms

$$\left[q^2 - m^2 + i\epsilon\right] = \left[q_0 + \omega - i\delta\right]\left[q_0 - \omega + i\delta\right] \tag{492}$$

$$\left[(p-q)^2 + i\epsilon\right] = \left[q_0 - p_0 + K - i\delta\right]\left[q_0 - p_0 - K + i\delta\right] , \tag{493}$$

where we define $\mathbf{K} = \mathbf{p} - \mathbf{q}$ with $K = |\mathbf{K}|$, $\omega = \sqrt{\mathbf{q}^2 + m^2}$ and $\delta > 0$ with the limit $\delta \to 0$ assumed. For contour integration, the poles in the upper half plane are clearly

$$q_0 = \begin{cases} -\omega + i\delta \\ p_0 - K + i\delta \end{cases} . \tag{494}$$

Thus integrating, we obtain

$$\Sigma(p) = \frac{e^2}{\hbar}\int \frac{d^3q}{(2\pi)^3}$$
$$\left\{\frac{(p_0-\omega)^2 - (\mathbf{p}+\mathbf{q})^2}{(-2\omega)\left((p_0+\omega)^2 - K^2\right)} + \frac{(2p_0-K)^2 - (\mathbf{p}+\mathbf{q})^2}{\left((p_0-K)^2 - \omega^2\right)(-2K)} - \frac{4}{(-2K)}\right\} . \tag{495}$$

We note that

$$\begin{aligned}
&\frac{1}{2\omega\left((p_0+\omega)^2 - K^2\right)} + \frac{1}{2K\left((p_0-K)^2 - \omega^2\right)} \\
&= \frac{1}{2\omega 2K}\left[\frac{1}{p_0-\omega-K} - \frac{1}{p_0+\omega-K}\right] + \frac{1}{2\omega 2K}\left[\frac{1}{p_0+\omega-K} - \frac{1}{p_0+\omega+K}\right] \\
&= -\frac{1}{2\omega 2K}\left[\frac{1}{(p_0+\omega+K)} + \frac{1}{(-p_0+\omega+K)}-\right] ,
\end{aligned} \tag{496}$$

and

$$\begin{aligned}
&\frac{(p_0-\omega)^2}{2\omega\left((p_0+\omega)^2 - K^2\right)} + \frac{(2p_0-K)^2}{2K\left((p_0-K)^2 - \omega^2\right)} \\
&= \frac{\omega^2 K + K^2\omega - p_0^2 K - 4p_0^2\omega}{2\omega K\,(p_0+\omega+K)\,(-p_0+\omega+K)} \\
&= \frac{1}{2K} - \frac{1}{2\omega 2K}\left[\frac{(p_0-\omega)^2}{(p_0+\omega+K)} + \frac{(p_0+\omega)^2}{(-p_0+\omega+K)}\right] .
\end{aligned} \tag{497}$$

Comparing these results with the expression (495) we can write the countert-erm

$$\Sigma(p) = \frac{e^2}{\hbar}\int \frac{d^3q}{(2\pi)^3 2\omega}\frac{1}{2K}$$
$$\left[6\omega + \frac{(p_0+\omega)^2 - (\mathbf{p}+\mathbf{q})^2}{(-p_0+\omega+K)} + \frac{(p_0-\omega)^2 - (\mathbf{p}+\mathbf{q})^2}{(p_0+\omega+K)}\right]. \quad (498)$$

Clearly, we may change the variable of integration from $\mathbf{q}$ to $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{V}(t)$. We then have $\omega \to E_q = \sqrt{\tilde{\mathbf{q}}^2 + m^2}$. The counterterm δm^2 is obtained by evaluating the self-energy $\Sigma(p)$ on the mass-shell i.e. with $p_0 = E_p$ and with $\mathbf{p} = \tilde{\mathbf{p}}$. It does not matter which point on the mass-shell is invoked because it is well known that the mass counterterm does not in fact explicitly depend on $\mathbf{p}$, but only on $p_0^2 - \mathbf{p}^2 = m^2 = E_p^2 - \tilde{\mathbf{p}}^2$. Technically, the mass-shell involves the physical mass m_P, however the counterterm δm^2 is itself is of order $\mathcal{O}(e^2)$ and for overall calculations at that order we may use m. Therefore we may see that the inner integral in (487) is nothing but $e^{-2}\delta m^2$, and thus independent of p, which shows that $\delta\mathcal{F} = -\mathcal{F}^0$ on comparison with (490) as stated. Consequently, the renormalised forward scattering amplitude does not contribute to the position shift.

CHAPTER 5

Quantum Green's Function Decomposition

> In this chapter we present an alternative derivation of some of the results for the position shift of the quantum scalar field based on the Green's function decomposition description of classical radiation reaction.

In the previous chapter we established that the classical position shift was reproduced in the $\hbar \to 0$ limit for the $\mathcal{O}(e^2)$ perturbation theory of quantum scalar electrodynamics. In fact, we showed that the position shift was entirely due to the emission process. Whilst giving equality between the two results our previous working does not however make clear any reasoning as for *why* this should come about. Given that the position shifts are equal, we may wish to know if the treatment of radiation reaction is the same in both theories. We may similarly ask what connections and differences there are between the two approaches with regards to the position shift. These questions are the subject of this chapter which we present as a short aside to the body of the work. That is not, however, to say that it is unimportant. On the contrary, here we present the clues gleaned mathematically from the results as to the interpretation of the quantum position shift contributions and the interpretation of the connections between classical and quantum theory with which we may view the body of work presented so far.

In order to attain these goals, we shall return to the earlier results for the scalar field and rework them to find expressions involving the Green's functions that were used in the classical derivation of the radiation reaction force. As such, we shall be using some of the model and results from the previous chapter and the appropriate descriptions and results shall be introduced again here when required.

We use the model of a wave packet of the scalar particle passing through a time-dependent potential for a finite period in the past of the measurements. Let the state of the wave packet of a scalar particle with momentum peaked about $\mathbf{p}$ be given by $|\mathbf{p}\rangle$. We recall that the final state for a particle undergoing radiation reaction is given to $\mathcal{O}(e^2)$ in our notation by

$$[1 + i\mathcal{F}(\mathbf{p})]|\mathbf{p}\rangle + \frac{i}{\hbar}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\hat{a}^\dagger_\mu(\mathbf{k})|\mathbf{P}\rangle\,, \tag{499}$$

with $k \equiv \|\mathbf{k}\|$ and $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$. We further recall that we found the position shift in the $\hbar \to 0$ limit to be given by

$$\delta x^i_{\mathrm{Q}} = -\frac{i}{2}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial}_{p^i} \mathcal{A}_\mu(\mathbf{p},\mathbf{k}) - \hbar\partial_{p^i}\mathrm{Re}\,\mathcal{F}(\mathbf{p})\,. \tag{500}$$

We approach the quantum emission and forward scattering processes and results in turn, starting with the former.

1. Emission decomposition

The emission contribution to the quantum position shift (500) is

$$\delta x^i_{\mathrm{em}} = -\frac{i}{2}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial}_{p^i} \mathcal{A}_\mu(\mathbf{p},\mathbf{k})\,. \tag{501}$$

We used the semiclassical approximation to find that the emission amplitude can be written in the $\hbar \to 0$ limit as

$$\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) = -e\int_{-\infty}^{+\infty} d\xi\, \frac{dx^\mu}{d\xi}\chi(\xi)\, e^{ik\xi}\,, \tag{360}$$

where we have included the cut-off function $\chi(\xi)$ that takes the value 1 when the external force is nonzero and smoothly becomes zero for large $|t|$ [1]. We now note that this expression for the emission amplitude coincides with that from a classical point charge to order $\hbar^0$.

$$\mathcal{A}^\mu_C(\mathbf{p},\mathbf{k}) = -\int d^4x\, e^{ik\cdot x} j^\mu(x)\,, \tag{502}$$

with the current $j^\mu(x)$ given by

$$j^\mu(x) = e\frac{dx^\mu}{dt}\delta^3(\mathbf{x} - \mathbf{X}_\mathbf{p}(t))\chi(t)\,, \tag{503}$$

[1]This expression was arrived at for both the time and space-dependent potentials.

where $\mathbf{X}_{\mathbf{p}}(t)$ is the path of the classical particle which passes through the origin with momentum $\mathbf{p}$. The classical field emitted from the current (503) is

$$A_{-}^{\mu}(x) = \int d^4x' G_{-\ \ \nu'}^{\ \mu}(x-x') j^{\nu'}(x') \,, \tag{504}$$

where G_{-} is the retarded Green's function. Now it is well known, and fairly straightforward to show, that the retarded Green's function can be written as [**24**]

$$G_{-\mu\nu'}(x-x') = i g_{\mu\nu'} \theta(t-t') \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \left[e^{-ik\cdot(x-x')} - e^{ik\cdot(x-x')} \right] . \tag{505}$$

These equations together imply that we can rewrite the classical (retarded) field in terms of the classical, and thus the quantum, emission amplitude i.e.

$$A_{-}^{\mu}(x) = -i \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \left[\mathcal{A}^{\mu}(\mathbf{p},\mathbf{k}) e^{-ik\cdot x} - \mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k}) e^{ik\cdot x} \right] , \tag{506}$$

for large enough t such that $\chi(t) = 0$. This gives us a fourier expansion of the classical field, written in terms of the quantum emission amplitude. We can consequently reverse this to rewrite the quantum emission amplitude in terms of the classical field. Firstly, we define the positive and negative frequency parts of the classical field as follows:

$$A^{(+)\mu} = -i \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \mathcal{A}^{\mu} e^{-ik\cdot x} \,, \tag{507}$$

$$A^{(-)\mu} = +i \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \mathcal{A}^{\mu *} e^{ik\cdot x} \,. \tag{508}$$

Note also that they are complex conjugates; $A^{(+)\mu *} = A^{(-)\mu}$. Inverting the fourier expansion of the field, we have the amplitude in terms of $\mathcal{A}^{(+)\mu}$

$$\mathcal{A}^{\mu} = 2k \int d^3\mathbf{x} A^{(+)\mu} e^{ik\cdot x} \,. \tag{509}$$

The position shift can thus be rewritten as follows

$$\begin{aligned} \delta x^i_{em} &= \frac{i}{2} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \left(2k \int d^3\mathbf{x}' A^{(-)\mu} e^{-ik\cdot x'} \right) \overleftrightarrow{\partial}_{p^i} \left(2k \int d^3\mathbf{x} A^{(+)}_{\mu} e^{ik\cdot x} \right) \\ &= \frac{i}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(\int d^3\mathbf{x}' A^{(-)\mu} e^{-ik\cdot x'} \right) \overleftrightarrow{\partial}_{p^i} \left(\int d^3\mathbf{x} A^{(+)}_{\mu} e^{ik\cdot x} \right) \\ &\quad \times \left\{ ik - (-ik) \right\} (-i) \,, \end{aligned}$$

(the last factor is just the remaining $2k$).

$$
\begin{aligned}
&= \frac{i}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(\int d^3\mathbf{x}' A^{(-)\mu} e^{-ik\cdot x'}\right)\left(-i\overleftrightarrow{\partial}_t\right)\overleftrightarrow{\partial}_{p^i}\left(\int d^3\mathbf{x} A^{(+)}_\mu e^{ik\cdot x}\right)\\
&= \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(\int d^3\mathbf{x}' A^{(-)\mu} e^{-ik\cdot x'}\right)\overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{p^i}\left(\int d^3\mathbf{x} A^{(+)}_\mu e^{ik\cdot x}\right)\\
&= \frac{1}{2}\int d^3\mathbf{x} A^{(-)\mu}\,\overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{p^i}\,A^{(+)}_\mu\,. \qquad (510)
\end{aligned}
$$

Now the position shift is real $\delta x^* = \delta x$ and also, due to the time derivative acting on the exponential in (507) and (508), we have

$$
\int d^3\mathbf{x} A^{(+)\mu}\,\overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{p^i}\,A^{(+)}_\mu = \int d^3\mathbf{x} A^{(-)\mu}\,\overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{p^i}\,A^{(-)}_\mu = 0\,. \qquad (511)
$$

Hence

$$
\begin{aligned}
\int d^3\mathbf{x} A^\mu_-\,\overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{p^i}\,A_{-\mu} &= \int d^3\mathbf{x}\left[A^{(+)\mu}\,\overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{p^i}\,A^{(-)}_\mu + A^{(-)\mu}\,\overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{p^i}\,A^{(+)}_\mu\right]\\
&= 2\int d^3\mathbf{x} A^{(-)\mu}\,\overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{p^i}\,A^{(+)}_\mu\,. \qquad (512)
\end{aligned}
$$

Thus

$$
\delta x^i_{em} = \frac{1}{4}\int d^3\mathbf{x} A^\mu_-\,\overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{p^i}\,A_{-\mu}\,. \qquad (513)
$$

Furthermore

$$
\begin{aligned}
A^\mu_-\,\overleftrightarrow{\partial}_t\overleftrightarrow{\partial}_{p^i}\,A_{-\mu} &= A^\mu_-\,\overleftrightarrow{\partial}_t\left(\partial_{p^i}A_{-\mu}\right) - \left(\partial_{p^i}A^\mu_-\right)\overleftrightarrow{\partial}_t\,A_{-\mu}\\
&= -\left(\partial_{p^i}A_{-\mu}\right)\overleftrightarrow{\partial}_t\,A^\mu_- - \left(\partial_{p^i}A^\mu_-\right)\overleftrightarrow{\partial}_t\,A_{-\mu}\\
&= -2\left(\partial_{p^i}A^\mu_-\right)\overleftrightarrow{\partial}_t\,A_{-\mu}\,. \qquad (514)
\end{aligned}
$$

Thus

$$
\delta x^i_{em} = -\frac{1}{2}\int d^3\mathbf{x}\left(\partial_{p^i}A^\mu_-\right)\overleftrightarrow{\partial}_t\,A_{-\mu}\,. \qquad (515)
$$

We now have an expression for the quantum position shift in terms of the retarded classical field.

1.1. Green's function substitution. We can use the Green's function decomposition of the classical field to rewrite the quantum position shift in terms of Green's functions instead of fields. Substituting the Green's function expression for the fields in the position shift, we obtain[2]

$$\begin{aligned}\delta x^i_{em} &= -\frac{1}{2}\int d^3\mathbf{x}\partial_{p^i}\left[\int d^4x' G_-^{\mu\nu'} j_{\nu'}\right]\overleftrightarrow{\partial}_t\left[\int d^4x'' G_{-\mu\rho''} j^{\rho''}\right]\\ &= -\frac{1}{2}\int d^4x' d^4x''\left[\int d^3\mathbf{x} G_-^{\mu\nu'}\overleftrightarrow{\partial}_t G_{-\mu\rho''}\right] j^{\rho''}\partial_{p^i} j_{\nu'}\,. \end{aligned}\tag{516}$$

We note that using the Kirchhoff representation, the regular Green's function can be written in terms of the retarded one, viz

$$G_{\mathrm{R}\alpha''\beta'}(x''-x') = -\frac{1}{2}\int_{t=T} d^3\mathbf{x}\, G_-{}^{\mu}{}_{\alpha''}(x-x'')\overleftrightarrow{\partial}_t G_{-\mu\beta'}(x-x')\,, \tag{517}$$

for $x_0 > \max(x'_0, x''_0)$. Substituting into (516) we obtain

$$\begin{aligned}\delta x_{\mathrm{em}} &= \int d^4x' d^4x''\,\partial_{p^i} j_{\rho''}(x'') G_R^{\rho''\nu'}(x''-x') j_{\nu'}(x')\\ &= \int d^4x\,\partial_{p^i} j^\mu(x) A_{\mathrm{R}\mu}(x)\,,\end{aligned}\tag{518}$$

where we have changed the notation slightly and identified the regular field generated by G_R.

The partial derivative acts on the current $j^\mu(x)$. Let $j_p(x)$ be the current following the path with final momentum p and let $j_{p+\Delta p}$ be the current following the path with final momentum $p+\Delta p$. This second path will be shifted from the original path, $\mathbf{X}$, by $\mathbf{\Delta X}$. Explicitly these currents can be written

$$j^\mu_p(x) = e\frac{dx^\mu}{dt}\delta^3(\mathbf{x}-\mathbf{X}) \tag{519}$$

$$j^\mu_{p+\Delta p}(x) = e\frac{dx^\mu}{dt}\delta^3(\mathbf{x}-\mathbf{X}-\mathbf{\Delta X})\,. \tag{520}$$

The derivative can then be written in limit form as

$$\partial_{p^i} j^\mu(t,\mathbf{x}) = \lim_{\Delta p^i\to 0}\frac{j^\mu_{p+\Delta p}(x) - j^\mu_p(x)}{\Delta p^i}\,. \tag{521}$$

[2]We have left off the arguments of the functions for brevity as they are obvious from the indices

Now, defining the four-vector $\Delta X^\alpha = (0, \mathbf{\Delta X})$, we write

$$\delta x^i_{\rm em} = \lim_{\Delta p^i \to 0} \frac{1}{\Delta p^i} \int d^4x \left[j^\mu_{p+\Delta p}(x) - j^\mu_p(x) \right] A_{{\rm R}\mu}(x) \,. \tag{522}$$

Substituting the explicit expressions for the currents, (519) and (520), we find the position shift as

$$\begin{aligned}
\delta x^i_{\rm em} &= \lim_{\Delta p^i \to 0} \frac{1}{\Delta p^i} \int d^4x \left[e\frac{dx^\mu}{dt}\delta^3(\mathbf{x} - \mathbf{X} - \mathbf{\Delta X}) - e\frac{dx^\mu}{dt}\delta^3(\mathbf{x} - \mathbf{X}) \right] A_{{\rm R}\mu}(x) \\
&= \lim_{\Delta p^i \to 0} \frac{e}{\Delta p^i} \int dt \left(\left(\frac{dX^\mu}{dt} + \frac{d\Delta X^\mu}{dt} \right) A_{{\rm R}\mu}((X + \Delta X) - \frac{dX^\mu}{dt} A_{{\rm R}\mu}(X) \right) \\
&= \lim_{\Delta p^i \to 0} \frac{e}{\Delta p^i} \int dt \left(\left(\frac{dX^\mu}{dt} + \frac{d\Delta X^\mu}{dt} \right) [A_{{\rm R}\mu}(X) + \Delta X^\alpha \nabla_\alpha A_{{\rm R}\mu}(X)] \right. \\
&\qquad\qquad \left. - \frac{dX^\mu}{dt} A_{{\rm R}\mu}(X) \right) \\
&= \lim_{\Delta p^i \to 0} \frac{e}{\Delta p^i} \int dt \left(\frac{dX^\mu}{dt} \Delta X^\alpha \nabla_\alpha A_{{\rm R}\mu}(X) + \frac{d\Delta X^\mu}{dt} A_{{\rm R}\mu}(X) \right) \,,
\end{aligned} \tag{523}$$

where due to the limit (recall $\Delta X \to 0$ as $\Delta p \to 0$) we need keep only the terms up to first order in ΔX. Integrating the second term of the integrand by parts[3] we obtain

$$\begin{aligned}
\delta x^i_{\rm em} &= \lim_{\Delta p^i \to 0} \frac{e}{\Delta p^i} \int dt \left(\frac{dX^\mu}{dt} \Delta X^\alpha \nabla_\alpha A_{{\rm R}\mu}(X) - \Delta X^\mu \frac{d}{dt} A_{{\rm R}\mu}(X) \right) \\
&= \lim_{\Delta p^i \to 0} \frac{e}{\Delta p^i} \int dt \left(\frac{dX^\mu}{dt} \Delta X^\alpha \nabla_\alpha A_{{\rm R}\mu}(X) - \Delta X^\mu \frac{dX^\alpha}{dt} \nabla_\alpha A_{{\rm R}\mu}(X) \right) \,.
\end{aligned} \tag{524}$$

Swapping the spacetime indices in the two sums in the second term, the position shift can be rewritten as

$$\begin{aligned}
\delta x^i_{\rm em} &= \lim_{\Delta p^i \to 0} e \int dt \, \frac{dX^\mu}{dt} \frac{\Delta X^\alpha}{\Delta p^i} \left[\nabla_\alpha A_{{\rm R}\mu}(X) - \nabla_\mu A_{{\rm R}\alpha}(X) \right] \\
&= e \int dt \, F^R_{\mu\alpha} \frac{dX^\mu}{dt} \left(\frac{\partial X^\alpha}{\partial p^i} \right)_t \\
&= - \int_{-\infty}^0 dt \, \mathcal{F}_{\rm LD}{}^j \left(\frac{\partial X^j}{\partial p^i} \right)_t \,.
\end{aligned} \tag{525}$$

---

[3]Recall that $A_{{\rm R}\mu}$ vanishes at the position of the charge when there is no acceleration.

where we recall from the introduction that the radiative electromagnetic field tensor is defined in terms of the regular/radiative field analogously to the standard field tensor and that the Lorentz-Dirac force is given as the Lorentz-force generated by this field (see (43) and (44)). We recognise this result as the classical position shift given by (279). This calculation is analogous to the derivation of the Lorentz force from the standard Lagrangian for a point charge in an external electromagnetic field (see, e.g. Ref. [**21**]). We have made the upper bound of the t-integration to $t = 0$ in the last line because ${\mathcal{F}_{\mathrm{LD}}}^j = 0$ for $t > 0$. Thus, we have shown that the contribution from the emission of a photon to the position shift agrees with the classical counterpart using the Green's function method.

2. Forward-Scattering decomposition

We now turn to the forward-scattering contribution to the position shift given by

$$\delta x^i_{\mathrm{for}} = -\hbar \partial_{p^i} \mathrm{Re}\, \mathcal{F}(\mathbf{p})\,. \tag{526}$$

We have shown that this contribution vanishes in the end. More precisely, the leading order terms of the real part of the forward-scattering amplitude are exactly canceled by the contribution from the mass counter-term, i.e. it is eliminated to order $\hbar^0$ by the mass renormalisation. Here we shall see that the field generated by the singular Green's function appears in the calculation of $\delta x^i_{\mathrm{for}}$. We recall that in the classical theory, this contribution to the field is, as the name implies, singular and is subsequently subtracted from the field in a process akin to the mass renormalisation.

The forward-scattering amplitude comes from the one-loop diagram shown in Fig. 5.1 and the additional loop diagram from the seagull vertex. For the contribution from the intermediate *particle* state (as opposed to *anti-particle* state), we divided the momentum integral for the virtual photon in this loop diagram into two parts; one with momentum $\hbar\|\mathbf{k}\|$ less than $\hbar^\alpha\lambda$ and the other with momentum larger than $\hbar^\alpha\lambda$, where α and λ are constants. We chose α

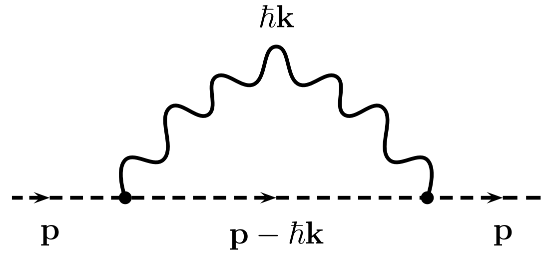

FIGURE 5.1. The one-loop diagram contributing to the forward-scattering amplitude: the dashed and wavy lines represent the scalar and photon propagators, respectively.

to satisfy $\frac{3}{4} < \alpha < 1$. However the condition $\alpha < 1$ will suffice for our current purpose. Denoting the first part with the virtual-photon momentum below the cut-off by $\mathcal{F}^<(\mathbf{p})$, we found that to lowest order in $\hbar$

$$\begin{aligned}\hbar\mathcal{F}^<(\mathbf{p}) = &- ie^2\int_{k\le\hbar^{\alpha-1}\lambda}\frac{d^3\mathbf{k}}{(2\pi)^3 2k}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}dt' \\ &\times\theta(t-t')\frac{dX^\mu_{\mathbf{p}}}{dt}\frac{dX_{\mathbf{p}\mu}}{dt'}e^{ik(t'-t)-i\mathbf{k}\cdot(\mathbf{X}_\mathbf{p}(t')-\mathbf{X}_\mathbf{p}(t))}\,. \end{aligned} \tag{527}$$

In the classical limit $\hbar\to 0$, the $\mathbf{k}$-integration will have no restriction because $\hbar^{\alpha-1}\lambda\to\infty$. As we did for the emission process, we can replace part of this expression with one containing Green's functions. Firstly, we note the presence of the classical currents and take advantage of the symmetry in the integrations to write

$$\begin{aligned}\hbar\mathcal{F}^<(\mathbf{p}) &= -\frac{1}{\hbar}\int d^4x d^4x' j^\mu(x) j^{\nu'}(x') i\hbar g_{\mu\nu'}\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k}\theta(t-t')e^{-ik\cdot(x-x')} \\ &= -\frac{1}{2\hbar}\int d^4x d^4x' j^\mu(x) j^{\nu'}(x') \\ &\quad\times i\hbar g_{\mu\nu'}\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k}\left[\theta(t-t')e^{-ik\cdot(x-x')}+\theta(t'-t)e^{-ik\cdot(x'-x)}\right]\,.\end{aligned} \tag{528}$$

Within this expression we recognize the form of the Feynman propagator, which can be given by

$$G_F^{\mu\nu'}(x-x') = -\,i\hbar g^{\mu\nu'} \int \frac{d^3k}{(2\pi)^3 2k} \left[\theta(t-t')e^{-ik\cdot(x-x')} + \theta(t'-t)e^{-ik\cdot(x'-x)}\right] . \tag{529}$$

We consequently find that in this limit

$$\hbar\mathcal{F}^{<}(\mathbf{p}) = \frac{1}{2\hbar} \int d^4x d^4x'\, j_\mu(x) j_{\nu'}(x') G_F^{\mu\nu'}(x-x') \,. \tag{530}$$

The contraction of the photon propagator with the external particle currents in the above expression is reminiscent of the one-loop diagram. However, it should be stressed that the above expression contracts this propagator with the *classical* currents and the validity is limited by the presence of both the $\hbar \to 0$ limit and the low-energy photon sector. To further our manipulation of the Green's functions, let us write the Feynman propagator as the sum of the real and imaginary parts:

$$G_F^{\mu\nu'}(x-x') = -\hbar G_S^{\mu\nu'}(x-x') - \frac{i\hbar}{2} G^{(1)\mu\nu'}(x-x') \,, \tag{531}$$

where we have Hadamard's elementary form, given by

$$\hbar G^{(1)\mu\nu'}(x-x') = \langle 0| \left\{ \hat{A}^\mu(x), \hat{A}^{\nu'}(x') \right\} |0\rangle \,, \tag{532}$$

with $\hat{A}^\mu(x)$ being the quantum electromagnetic potential. We thus spot the presence of the singular Green's function in our calculation. Before returning to this point, we briefly look at the imaginary part. Above the cut-off, there is no imaginary contribution in the $\hbar \to 0$ limit. By unitarity, $\mathrm{Im}\,\mathcal{F}^{<}(\mathbf{p})$ is required to equal half the emission probability. This has previously been shown to be the case by direct computation. It can also easily be shown from the above expression (530) using (531), (532) and the symmetry of the integration

and anticommutator:

$$\begin{aligned}\mathrm{Im}\,\mathcal{F}^{<}(\mathbf{p}) &= -\frac{1}{2\hbar^2}\int d^4xd^4x'j_\mu(x)\langle 0|\hat{A}^\mu(x)\hat{A}^{\nu'}(x')|0\rangle j_{\nu'}(x') \\ &= -\frac{1}{2\hbar^2}\int d^4xd^4x'j_\mu(x)j_{\nu'}(x')e^{-ik\cdot(x-x')} \\ &= -\frac{1}{2\hbar}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\,. \end{aligned} \tag{533}$$

Returning now to the real part we have

$$\hbar\mathrm{Re}\,\mathcal{F}^{<}(\mathbf{p}) = -\frac{1}{2}\int d^4xd^4x'\,j_\mu(x)j_\nu(y)G_S^{\mu\nu}(x-y)\,. \tag{534}$$

Using the symmetry of $G_S^{\mu\nu'}(x-x')$, we obtain

$$-\hbar\partial_{p^i}\mathrm{Re}\,\mathcal{F}^{<}(\mathbf{p}) = \int d^4x\,\partial_{p^i}j^\mu(x)A_{S\mu}(x)\,, \tag{535}$$

where $A_S^\mu(x)$ is the singular part of the self-field given by

$$A_S^\mu(x) = \int d^4x'\,G_S^{\mu\nu'}(x-x')j_{\nu'}(x')\,. \tag{536}$$

Equation (535) is analogous to the expression for the emission contribution to the position shift (518), which was in terms of the regular field. If we add (535) to the emission contribution (518), then

$$\delta x_{\mathrm{em}} - \hbar\partial_{p^i}\mathrm{Re}\,\mathcal{F}^{<}(\mathbf{p}) = \int d^4x\,\partial_{p^i}j_\mu(x)A_-^\mu(x)\,, \tag{537}$$

where the self-field, $A_-^\mu(x)$, is given by (504). Thus, the one-photon emission process and the low-energy part of the forward-scattering process are incorporated in the classical self-field $A_-^\mu(x)$ if one sees this field from the viewpoint of quantum derivation of the self-force. The regular part, $A_R^\mu(x)$, of the self-field in classical electrodynamics corresponds to the emission process in QED and the singular part, $A_S^\mu(x)$, to the low-energy forward-scattering process. The remaining high-energy and intermediate anti-particle state contributions to the forward-scattering amplitude in QED have no classical counterpart. The forward-scattering contribution as a whole vanishes if one includes the quantum mass counter-term, as was shown in Chapter 4.

CHAPTER 6

Spinor Quantum Position Shift

In this chapter we repeat our derivations and calculations for the quantum position shift using the canonical theory of quantum electrodynamics based on the model of the Dirac spinor field. We again combine the effects of the photon emission, forward scattering and mass renormalisation in the $\hbar \to 0$ limit in order to compare the result with the classical theory.

In this chapter we shall replace the scalar quantum field model with the more realistic spinor field of quantum electrodynamics in order to calculate the quantum position shift. We shall thus start our quantum position shift calculations from scratch using the spinor field definitions and the spinor semi-classical expansions from Chapter 1 section 8 and Chapter 2 section 3 respectively. Much of the path that we shall tread here will be familiar from the scalar work and some of the expressions derived from this source will be the same as before. Naturally, the classical position shift is unchanged, but we note that the classical theory does not include the concept of spin. Despite the similarities with our previous scalar work, there are however differences due to the construction of the fields, not least the addition of spin to consider in the interactions and evolutions. Whilst using the same approach as before, we shall nonetheless tread carefully and repeat most of the calculations from the new spinor particle definitions. For the potential, we shall look at the time-dependent (and spatially independent) case $V(t)$ throughout and thus take advantage of the conservation of momentum.

1. Initial control state

We again start with the expressions describing the initial control state, but this time for a spinor particle. As per the introduction to the spinor field definitions in Chapter 1 section 8, we define the initial incoming wave packet of the spinor field with spin labeled by α as

$$|i\rangle = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \sqrt{\frac{m}{p_0}} f(\mathbf{p}) b^\dagger_\alpha(\mathbf{p})|0\rangle \,, \tag{103}$$

where we recall that f is sharply peaked about the initial momentum in the region $\mathcal{M}_-$ and normalised via $\langle i|i\rangle = 1$, viz

$$\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}) f(\mathbf{p}) = 1 \,. \tag{104}$$

The spinor initial state differs from the scalar state only in the presence of the spinor field creation operator, with spin index, and also the factor multiplying the basic Lorentz-invariant measure: $1/(2p_0) \to m/p_0$, which is due to the canonical convention chosen for these fields. The outgoing wave packet of our control particle, which we recall does not undergo radiation reaction in $\mathcal{M}_I$, is given by the same expression, albeit with f now sharply peaked about the final momentum $\bar{\mathbf{p}}$ in the region $\mathcal{M}_+$ and α now represents the spin of the outgoing state. As with the scalar field, we let the potential satisfy $|V_0| < 2m$, thus precluding the possibility of particle pair creation. The associated vacuum effects can then be safely ignored and the charge density can be considered equivalent to the probability density for a one-particle state. The expectation

value of the density of the state $|i\rangle$, $\langle\rho(x)\rangle = \langle i| :\psi^\dagger\psi: |i\rangle$ is given as follows

$$
\begin{aligned}
&\langle\rho(x)\rangle \\
&= \langle 0|\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\sqrt{\frac{m}{p'_0}}f^*(\mathbf{p}')b_\alpha(\mathbf{p}') \\
&\quad :\left[\int \frac{d^3\mathbf{p}''}{(2\pi\hbar)^3}\frac{m}{p''_0}\sum_\beta b^\dagger_\beta(\mathbf{p}'')\Phi^\beta(p'')\int \frac{d^3\mathbf{p}'''}{(2\pi\hbar)^3}\frac{m}{p'''_0}\sum_\gamma b_\gamma(\mathbf{p}''')\Phi^{\gamma\dagger}(p''')\right]: \\
&\quad \times \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\sqrt{\frac{m}{p_0}}f(\mathbf{p})b^\dagger_\alpha(\mathbf{p})|0\rangle \\
&= \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}''}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}'''}{(2\pi\hbar)^3}\frac{m}{\sqrt{p_0p'_0}}\frac{m^2}{p''_0p'''_0}f^*(\mathbf{p}')f(\mathbf{p})\sum_{\beta,\gamma}\Phi^{\beta\dagger}(p'')\Phi^\gamma(p''') \\
&\quad \times \langle 0|b_\alpha(\mathbf{p}')b^\dagger_\beta(\mathbf{p}'')b_\gamma(\mathbf{p}''')b^\dagger_\alpha(\mathbf{p})|0\rangle \\
&= \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}''}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}'''}{(2\pi\hbar)^3}\frac{m}{\sqrt{p_0p'_0}}\frac{m^2}{p''_0p'''_0}f^*(\mathbf{p}')f(\mathbf{p})\sum_{\beta,\gamma}\Phi^{\beta\dagger}(p'')\Phi^\gamma(p''') \\
&\quad \times \frac{\mathbf{p}''_0}{m}(2\pi\hbar)^3\delta^3(\mathbf{p}'-\mathbf{p}'')\delta_{\alpha\beta}\frac{\mathbf{p}'''_0}{m}(2\pi\hbar)^3\delta^3(\mathbf{p}'''-\mathbf{p})\delta_{\alpha\gamma} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{m}{\sqrt{p_0p'_0}}f^*(\mathbf{p}')f(\mathbf{p})\Phi^\dagger_\alpha(\mathbf{p}')\Phi_\alpha(\mathbf{p})\,. \qquad (538)
\end{aligned}
$$

We wish to measure the position expectation value at time $t=0$. As this lies, by definition, far into the region $\mathcal{M}_+$ we may use the mode functions for the free field i.e. $\Phi_\alpha(\mathbf{p}) = u_\alpha(p)e^{ip\cdot x/\hbar}$. Hence

$$
\begin{aligned}
\langle x^i(0)\rangle &= \int d^4x\, x^i\langle i| :\psi^\dagger\psi: |i\rangle \\
&= \int d^3x \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{m}{\sqrt{p_0p'_0}}f^*(\mathbf{p}')f(\mathbf{p})u^\dagger_\alpha(p')u_\alpha(p)e^{-i(\mathbf{p}-\mathbf{p}')\cdot x/\hbar}x^i \\
&= \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{m}{\sqrt{p_0p'_0}}f^*(\mathbf{p}')f(\mathbf{p})u^\dagger_\alpha(p')u_\alpha(p) \\
&\quad\times \int d^3x(-i\hbar)\partial_{p_i}e^{-i(\mathbf{p}-\mathbf{p}')\cdot x}\,.
\end{aligned}
$$

Integrating by parts, and integrating out the resultant delta function, we obtain

$$
\langle x^i(0)\rangle = i\hbar\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}m\left(\frac{f^*(\mathbf{p})}{\sqrt{p_0}}u^\dagger_\alpha(p)\right)\partial_{p_i}\left(\frac{f(\mathbf{p})}{\sqrt{p_0}}u_\alpha(p)\right)\,. \qquad (539)
$$

This expression will in fact be the one we need to recall when comparing with the final interacting state. However, if we complete the p^i differentiation, the resulting terms turn out to be of different orders in $\hbar$. Although we are only dealing with the $\hbar \to 0$ limit, for completeness we shall differentiate and analyse these terms further: The term differentiated with respect to the momentum in the x^i direction is

$$\partial_{p_i}\left(\frac{f(\mathbf{p})}{\sqrt{p_0}}u_\alpha(p)\right) = \frac{\partial_{p_i}f(\mathbf{p})}{\sqrt{p_0}}u_\alpha(p) - \frac{f(\mathbf{p})p_i}{2p_0^{5/2}}u_\alpha(p) + \frac{f(\mathbf{p})}{\sqrt{p_0}}\partial_{p_i}u_\alpha(p)\,, \tag{540}$$

thus giving the position expectation value as

$$i\hbar\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{m}{p_0}$$
$$\left\{\,[f^*(\mathbf{p})\partial_{p_i}f(\mathbf{p})]\,u_\alpha^\dagger(p)u_\alpha(p) - |f(\mathbf{p})|^2\frac{p_i}{2p_0^2}u_\alpha^\dagger(p)u_\alpha(p) + |f(\mathbf{p})|^2 u_\alpha^\dagger(p)\partial_{p_i}u_\alpha(p)\right\}. \tag{541}$$

From the definition of the free spinor in (101)

$$u_\alpha(p) = \sqrt{\frac{p_0+m}{2m}}\begin{pmatrix} s_\alpha \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}s_\alpha\end{pmatrix}\,, \tag{101}$$

we have the normalisation $u_\alpha^\dagger(p)u_\alpha(p) = p_0/m$. We now calculate the product $u_\alpha^\dagger(p)\partial_{p_i}u_\alpha(p)$. The momentum derivatives of the two factors in the spinor (101) are

$$\partial_{p_i}\left(\sqrt{\frac{p_0+m}{2m}}\right) = \frac{1}{2}\frac{1}{\sqrt{2m\,(p_0+m)}}\frac{p_i}{p_0} \tag{542}$$

$$\partial_{p_i}\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\sqrt{2m\,(p_0+m)}}\right) = \frac{\boldsymbol{\sigma}\cdot\mathbf{n_i}}{\sqrt{2m\,(p_0+m)}} - \frac{1}{2}\frac{1}{\sqrt{2m}}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{(p_0+m)^{3/2}}\frac{p_i}{p_0}\,, \tag{543}$$

where $\mathbf{n_i}$ is the unit vector is the ith direction.

The general expression $u_\alpha^\dagger \partial_{p_i} u_\beta$ is thus given by

$$
\begin{aligned}
&u_\alpha^\dagger(p)\partial_{p_i}u_\beta(p)\\
&=\left(\sqrt{\frac{p_0+m}{2m}}\,s_\alpha^\dagger\right)\left(\frac{1}{\sqrt{2m\,(p_0+m)}}\frac{p_i}{2p_0}s_\beta\right)\\
&\quad+\left(\sqrt{\frac{p_0+m}{2m}}\,s_\alpha^\dagger\,\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{p_0+m}\right)\left(\frac{\boldsymbol{\sigma}\cdot\mathbf{n_i}}{\sqrt{2m\,(p_0+m)}}\,s_\beta-\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{(p_0+m)^{3/2}}\frac{p_i}{2p_0\sqrt{2m}}\,s_\beta\right)\\
&=\frac{1}{2}\frac{p_i}{2mp_0}s_\alpha^\dagger s_\beta+\frac{s_\alpha^\dagger\left(\boldsymbol{\sigma}\cdot\mathbf{p}\,\boldsymbol{\sigma}\cdot\mathbf{n_i}\right)s_\beta}{2m(p_0+m)}-\frac{1}{2}\frac{p_i}{2mp_0}\frac{s_\alpha^\dagger\left(\boldsymbol{\sigma}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}\right)s_\beta}{(p_0+m)^2}\\
&=\frac{1}{2}\frac{p_i}{2mp_0}s_\alpha^\dagger s_\beta+\frac{s_\alpha^\dagger\left(p_i+i(\mathbf{p}\times\mathbf{n_i})\cdot\boldsymbol{\sigma}\right)s_\beta}{2m(p_0+m)}-\frac{1}{2}\frac{p_i}{2mp_0}\frac{s_\alpha^\dagger\mathbf{p}^2 s_\beta}{(p_0+m)^2}\\
&=\frac{1}{2}\frac{p_i}{2mp_0}s_\alpha^\dagger s_\beta+\frac{p_i}{2m(p_0+m)}s_\alpha^\dagger s_\beta+\frac{is_\alpha^\dagger\mathbf{n_i}\cdot\left(\boldsymbol{\sigma}\times\mathbf{p}\right)s_\beta}{2m(p_0+m)}\\
&\quad-\frac{1}{2}\frac{p_i}{2mp_0}\frac{(p_0^2-m^2)}{(p_0+m)^2}s_\alpha^\dagger s_\beta\\
&=\frac{p_i}{2mp_0}s_\alpha^\dagger s_\beta+\frac{i}{2m(p_0+m)}s_\alpha^\dagger\mathbf{n_i}\cdot\left(\boldsymbol{\sigma}\times\mathbf{p}\right)s_\beta\,,
\end{aligned}
\tag{544}
$$

where we have used $\boldsymbol{\sigma}\cdot\mathbf{a}\,\boldsymbol{\sigma}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{b}I+i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}$ and $\mathbf{p}\cdot\mathbf{p}=p_0^2-m^2$. We consequently obtain

$$
u_\alpha^\dagger\partial_{p_i}u_\beta=\frac{p_i}{2mp_0}\delta_{\alpha\beta}+\frac{i}{2m(p_0+m)}s_\alpha^\dagger\mathbf{n_i}\cdot\left(\boldsymbol{\sigma}\times\mathbf{p}\right)s_\beta\,. \tag{545}
$$

As a short aside, we can look at an interpretation of the second term in (545). In the case where $\alpha=\beta$, as we require, then $s_\alpha^\dagger\mathbf{n_i}\cdot\left(\boldsymbol{\sigma}\times\mathbf{p}\right)s_\alpha=\mathbf{n_i}\cdot\boldsymbol{\xi}\times\mathbf{p}$ where $\boldsymbol{\xi}$ is the unit vector in the direction of the spin (positive for spin up, negative for spin down). This term is an example of the effects of the addition of spin to the quantum model, in this case on the measurement of the position in the i direction. Now, the expression measures the component in the i direction of the vector $\boldsymbol{\xi}\times\mathbf{p}$, perpendicular to the spin and the momentum (and is zero when these coincide)[1] and could be described as providing a change, in the momentum and consequently the position, due to the interaction between the momentum and the spin. This type of effect can be seen by analysing the

[1]The term is also zero in the direction of either the spin or the momentum.

Poincaré algebra of the generators for a boost and a rotation. Note that the same algebra is obeyed by the spin and boost operators for a spinor. If K_i represents the generator of a boost in the i direction and J_j represents the generator of a rotation in the j direction then we have

$$[K_i, K_j] = -i\epsilon_{ijk} J_k \,, \tag{546}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \,, \tag{547}$$

where ϵ_{ijk} is the Levi-Civita symbol. The first relation leads to the Thomas precession correction to the spin-orbit interaction. The second relation is related to the current effect. This term is naturally not present for calculations using the scalar field. Our extra term is thus a mathematical consequence of the fact that the spin and boost operators do not commute.

Returning to our main calculation, the position expectation value (at $t = 0$) of the initial state can be given as

$$\begin{aligned} & i\hbar \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} m \left(\frac{f^*(\mathbf{p}) u_\alpha^\dagger(p)}{\sqrt{p_0}}\right) \partial_{p_i} \left(\frac{f(\mathbf{p}) u_\alpha(p)}{\sqrt{p_0}}\right) \\ = {} & i\hbar \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \Big\{ m f^*(\mathbf{p}) \partial_{p_i} f(\mathbf{p}) \frac{1}{p_0} \frac{p_0}{m} \\ & + m|f(\mathbf{p})|^2 \left(-\frac{p_i}{2p_0^3} \frac{p_0}{m} + \frac{1}{p_0} \left[\frac{p_i}{2mp_0} + \frac{i\mathbf{n_i} \cdot \boldsymbol{\xi} \times \mathbf{p}}{2m(p_0 + m)} \right] \right) \Big\} \\ = {} & i\hbar \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \left\{ f^*(\mathbf{p}) \partial_{p_i} f(\mathbf{p}) + |f(\mathbf{p})|^2 \frac{i\mathbf{n_i} \cdot \boldsymbol{\xi} \times \mathbf{p}}{2p_0(p_0 + m)} \right\} . \end{aligned} \tag{548}$$

Because, after a $(2\pi\hbar)^3$ pre-multiple, $f^*(\mathbf{p})\partial_p f(\mathbf{p})$ is of order $\hbar^{-1}$, whereas $|f(\mathbf{p})|^2$ is $\mathcal{O}(\hbar^0)$, the second term in (548) is of order $\hbar$ and in the classical limit the spin-related effect given above does not contribute. In this limit we have

$$\langle x^i \rangle|_{t=0} = i\hbar \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}) \partial_{p_i} f(\mathbf{p}) \,. \tag{549}$$

We recognize this expression as the same as that was reached for the scalar field in (286). Once again, as the position shift is real, we may write

$$\langle x^i \rangle|_{t=0} = \frac{i\hbar}{2} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}) \overleftrightarrow{\partial}_{p_i} f(\mathbf{p}) \,. \tag{550}$$

2. Final interacting state

For the final state of a particle undergoing radiation reaction, we again start with an incoming wave packet of the form $|i\rangle$. The interactions to order e^2 for the spinor field are, as with the scalar field, composed of the photon emission sector and a one-loop forward scattering sector along of course with the null interaction. We shall use notation similar to the scalar field for the amplitudes of these processes. These amplitudes we shall of course later calculate (in sections 3 and 4 of this chapter) using the semiclassical spinor expansions for the interacting region $\mathcal{M}_I$, whilst the measurement of the position shift takes place at $t = 0$ inside $\mathcal{M}_+$. Let us start with the final state giving the definitions of the amplitudes:

$$[1 + i\mathcal{F}(\mathbf{p})]b^\dagger_\alpha(\mathbf{p})|0\rangle + \frac{i}{\hbar}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{(\beta)\mu}_{(\alpha)}(\mathbf{p},\mathbf{k})a^\dagger_\mu(\mathbf{k})b^\dagger_\beta(\mathbf{p}')|0\rangle\,, \tag{551}$$

where $\mathcal{F}$ represents the forward scattering amplitude from the one loop self-interaction and $\mathcal{A}$ represents the amplitude from the one-photon emission. In the case of the forward scattering and non-interacting processes, the spin and momentum of the final states are the same. Note however, that for the one-photon emission this is no longer the case. The final momentum is labelled $\mathbf{p}'$ here and the final spin β. The emission amplitude thus contains two spin indices. Nevertheless, to lowest order in $\hbar$ the spin does not change. In addition, by momentum conservation, the final momentum is equal to $\mathbf{p} - \hbar\mathbf{k}$ which we shall label $\mathbf{P}$. These relations will be proved later when we explicitly calculate the emission amplitude, but we shall utilize them now in order to simplify the following calculations and drop the spin indices on the emission amplitude. Let us thus define the following parts of the final state

$$|f\rangle_{\text{for}} = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\sqrt{\frac{m}{p_0}}\,[1 + i\mathcal{F}(\mathbf{p})]\, f(\mathbf{p})b^\dagger_\alpha(\mathbf{p})|0\rangle \tag{552}$$

$$|f\rangle_{\text{em}} = \frac{i}{\hbar}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\sqrt{\frac{m}{p_0}}\mathcal{A}^\mu(\mathbf{k},\mathbf{p})f(\mathbf{p})a^\dagger_\mu(\mathbf{k})b^\dagger_\alpha(\mathbf{P})|0\rangle\,. \tag{553}$$

As there will be no cross-term between these two states, the final state density is the sum of the densities of the above two state. We now proceed to calculate these densities and consequently obtain an expression for the position expectation value of the final state in terms of the two amplitudes. These calculations follow those from the scalar field very closely and the reader may wish to refer to them.

2.1. Zero photon sector. The zero photon sector density is given by

$$ {}_{\rm for}\langle f| : \psi^\dagger \psi : |f\rangle_{\rm for} \,. \tag{554} $$

The state $|f\rangle_{\rm for}$ is nearly identical to that used for the scalar case and again note that the calculation of the density is identical to that for the non-interacting state with the substitution $f(\mathbf{p}) \to [1 + i\mathcal{F}(\mathbf{p})]\, f(\mathbf{p})$. The density is thus

$$ \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3} \frac{m}{\sqrt{p_0 p_0'}} \left[1 - i\mathcal{F}^*(\mathbf{p}')\right] f^*(\mathbf{p}') \left[1 + i\mathcal{F}(\mathbf{p})\right] f(\mathbf{p}) \Phi^{\alpha\dagger}(\mathbf{p}')\Phi^\alpha(\mathbf{p}) \,. \tag{555} $$

The $t = 0$ position expectation value of the $|f\rangle_{for}$ state is hence

$$ \begin{aligned} \langle x^i(0)\rangle_{\rm for} = \frac{i\hbar}{2} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} m \\ \times \left([1 - i\mathcal{F}^*(\mathbf{p})] \frac{f^*(\mathbf{p})}{\sqrt{p_0}} u^\dagger(p) \right) \overleftrightarrow{\partial}_{p_i} \left([1 + i\mathcal{F}(\mathbf{p})] \frac{f(\mathbf{p})}{\sqrt{p_0}} u(p) \right) . \end{aligned} \tag{556} $$

Expanding out the terms to order e^2 (i.e. ignoring the $\mathcal{F}^*\mathcal{F}$ type terms) we have

$$ \begin{aligned} \langle x^i(0)\rangle_{\rm for} = \frac{i\hbar}{2} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} m \left(\frac{f^*(\mathbf{p}) u^\dagger(p)}{\sqrt{p_0}} \right) \overleftrightarrow{\partial}_{p_i} \left(\frac{f(\mathbf{p}) u(p)}{\sqrt{p_0}} \right) [1 - 2\Im\mathcal{F}] \\ - \hbar \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} |f(\mathbf{p})|^2 \partial_{p_i} \Re\mathcal{F}(\mathbf{p}) \,. \end{aligned} \tag{557} $$

The expression obtained is analogous to the scalar case in that we find the appropriate form of the non-interacting position combined with $[1 - 2\Im\mathcal{F}]$

which we shall deal with later, and a further term dependent on the real part of the forward scattering amplitude.[2]

2.2. One photon sector. The density and position expectation value for the one photon sector is more complicated due to the fact that the final state electron is now moving with momentum $\mathbf{P}$ rather than $\mathbf{p}$. We dealt with this problem before with the scalar field, and as we are dealing with a time-dependent potential, we may make use of the conservation of momentum.[3] The density is given by

$$\begin{aligned}
&{}_{\mathrm{em}}\langle f| : \psi^\dagger \psi : |f\rangle_{\mathrm{em}} \\
&= \frac{-i}{\hbar} \int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3 2k_0'} \sqrt{\frac{m}{p_0'}} \mathcal{A}^{\nu\dagger}(\mathbf{k}',\mathbf{p}') f(\mathbf{p}') \\
&\quad \times \frac{i}{\hbar} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} \sqrt{\frac{m}{p_0}} \mathcal{A}^{\mu}(\mathbf{k},\mathbf{p}) f(\mathbf{p}) \\
&\quad \times \langle 0| a_\nu(\mathbf{k}') b_\alpha(\mathbf{P}') : \psi^\dagger \psi : a_\mu^\dagger(\mathbf{k}) b_\alpha^\dagger(\mathbf{P}) |0\rangle \\
&= -\frac{1}{\hbar} \int \frac{d^3\mathbf{p}' d^3\mathbf{p}}{(2\pi\hbar)^6 \sqrt{p_0 p_0'}} \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} m \Big(f^*(\mathbf{p}') \mathcal{A}_\mu^*(\mathbf{p}',\mathbf{k}) \Phi_\alpha^\dagger(\mathbf{P}') \Big) \Big(f(\mathbf{p}) \mathcal{A}^\mu(\mathbf{p},\mathbf{k}) \Phi_\alpha(\mathbf{P}) \Big) ,
\end{aligned} \tag{558}$$

where $\mathbf{P}' = \mathbf{p}' - \hbar\mathbf{k}'$ and we have used the anticommutation relations for spinor field (95), as used for the initial control particle calculations, and the commutation relations for the electromagnetic field (83). Note that the mode functions present in this expression are those of the free field for the density

[2]In fact, careful analysis of the calculation would show that the momentum derivative of the imaginary part of $\mathcal{F}$ is necessarily zero.

[3]The reader may recall that the time-dependent case is slightly more straight forward in this respect than that for the space-dependent case.

in the $\mathcal{M}_+$ region. The position expectation value (at $t = 0$) is therefore

$$
\begin{aligned}
&\langle x^i(0)\rangle_{\rm em} \\
&= -\frac{1}{\hbar}\int \frac{d^3\mathbf{p}' d^3\mathbf{p} m}{(2\pi\hbar)^6\sqrt{p_0 p_0'}}\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\Big(f^*(\mathbf{p}')\mathcal{A}^*_\mu(\mathbf{p}',\mathbf{k})u^\dagger_\alpha(P')\Big)\Big(f(\mathbf{p})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})u_\alpha(P)\Big) \\
&\quad\times\int d^3x\partial_{P_i}\left(i\hbar e^{-i(\mathbf{P}-\mathbf{P}'\cdot\mathbf{x})/\hbar}\right) \\
&= -i\int\frac{d^3\mathbf{p}' d^3\mathbf{p}}{(2\pi\hbar)^6}\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}m \\
&\quad\times\Big(\frac{f^*(\mathbf{p}')}{\sqrt{p_0'}}\mathcal{A}^*_\mu(\mathbf{p}',\mathbf{k})u^\dagger_\alpha(P')\Big)\frac{\partial}{\partial P_i}\Big(\frac{f(\mathbf{p})}{\sqrt{p_0}}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})u_\alpha(P)\Big)(2\pi\hbar)^3\delta^3(\mathbf{P}-\mathbf{P}')\,.
\end{aligned}
\tag{559}
$$

Given the definitions of $\mathbf{P}$ and $\mathbf{P}'$, we have $\delta^3(\mathbf{P}-\mathbf{P}') = \delta^3(\mathbf{p}-\mathbf{p}')$. The position shift due to emission is therefore given by the expression

$$
\begin{aligned}
\langle x^i(0)\rangle_{\rm em} &= -\frac{i}{2}\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}m \\
&\qquad \left(\frac{f^*(\mathbf{p})}{\sqrt{p_0}}\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})u^\dagger_\alpha(P)\right)\overleftrightarrow{\partial}_{P_i}\left(\frac{f(\mathbf{p})}{\sqrt{p_0}}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})u_\alpha(P)\right).
\end{aligned}
\tag{560}
$$

This can be split into two parts:

$$
\begin{aligned}
&\langle x^i(0)\rangle_{\rm em} \\
&= -\frac{i}{2}\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3 p_0}\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}m\left(\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\overleftrightarrow{\partial}_{P_i}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\right)|f(\mathbf{p})|^2 u^\dagger_\alpha(P)u_\alpha(P) \\
&\quad -\frac{i}{2}\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}m\left(\frac{f^*(\mathbf{p})}{\sqrt{p_0}}u^\dagger_\alpha(P)\right)\overleftrightarrow{\partial}_{P_i}\left(\frac{f(\mathbf{p})}{\sqrt{p_0}}u_\alpha(P)\right) \\
&\qquad\times\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\,.
\end{aligned}
\tag{561}
$$

The first integral gives to lowest order

$$
\begin{aligned}
&-\frac{i}{2}\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\left(\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\overleftrightarrow{\partial}_{p_i}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\right)\frac{P_0}{p_0}\frac{\partial p_i}{\partial P_i} \\
&= -\frac{i}{2}\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\left(\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\overleftrightarrow{\partial}_{p_i}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\right),
\end{aligned}
\tag{562}
$$

where in the last line we have made use of the fact that to order $\hbar^0$, $P_0 = p_0$ and $\partial p_i/\partial P_i = 1$.

2.3. Unitarity. In the scalar calculation, we gained a further relation between the imaginary part of the forward scattering and the emission probability using the normalisation of the final state, viz

$$\langle f|f\rangle = 1\,. \tag{563}$$

We can complete the same calculation again and find that we do in fact find the same relation. The final state is given by the sum of $|f\rangle_{\text{for}}$ and $|f\rangle_{\text{em}}$, in (552) and (553) respectively. The left hand side of (563) above is thus,

$$\begin{aligned}
&\langle f|f\rangle \\
&= \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\sqrt{\frac{m}{p_0'}}\sqrt{\frac{m}{p_0}} f^*(\mathbf{p})\,[1 - i\mathcal{F}^*(\mathbf{p})]\,[1 + i\mathcal{F}(\mathbf{p}')]\,f(\mathbf{p}')\langle 0|b_\alpha(\mathbf{p})b_\alpha^\dagger(\mathbf{p}')|0\rangle \\
&\quad + \frac{1}{\hbar^2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\frac{d^3\mathbf{k}'}{(2\pi)^3 2k_0'}\sqrt{\frac{m}{p_0'}}\sqrt{\frac{m}{p_0}} \\
&\quad \times f^*(\mathbf{p})\mathcal{A}^{*\mu}(k,\mathbf{p})f(\mathbf{p}')\mathcal{A}^\nu(k',\mathbf{p}')\langle 0|b_\alpha(\mathbf{P})a_\mu(\mathbf{k})a_\nu^\dagger(\mathbf{k}')b_\alpha^\dagger(\mathbf{P}')|0\rangle\,.
\end{aligned} \tag{564}$$

We again make use of the conservation of momentum, with

$$\frac{P_0}{m}\delta^3(\mathbf{P}-\mathbf{P}') = \frac{p_0}{m}\delta^3(\mathbf{p}-\mathbf{p}')\frac{P_0}{p_0}\,. \tag{565}$$

Hence to order e^2 (i.e. only up to first order in $\mathcal{F}$) we have

$$\begin{aligned}
\langle f|f\rangle &= \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\,(1 - 2\Im\mathcal{F}(\mathbf{p})) \\
&\quad - \frac{1}{\hbar}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}|f(\mathbf{p})|^2\mathcal{A}_\mu^*(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\frac{P_0}{p_0}\,.
\end{aligned} \tag{566}$$

As this is equal to 1 (by (563)) and $f(\mathbf{p})$ is normalised by (104), we obtain

$$\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2 2\Im\mathcal{F}(\mathbf{p}) = -\frac{1}{\hbar}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\mathcal{A}_\mu^*(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\frac{P_0}{p_0}\,, \tag{567}$$

as before. Consequently, using the delta function limit for $|f(\mathbf{p})|^2$,

$$2\Im\mathcal{F}(\mathbf{p}) = -\frac{1}{\hbar}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\mathcal{A}_\mu^*(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\frac{P_0}{p_0}\,, \tag{568}$$

where we have relabeled the final peak momentum $\bar{\mathbf{p}} \to \mathbf{p}$.

2.4. Position of the final state. If we add the contributions to the position expectation value from the forward scattering and emission sectors we obtain

$$
\begin{aligned}
&\langle x^i(0)\rangle \\
&= \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} m \left(\frac{f^*(\mathbf{p})u^\dagger(p)}{\sqrt{p_0}}\right) \overleftrightarrow{\partial}_{p_i} \left(\frac{f(\mathbf{p})u(p)}{\sqrt{p_0}}\right) [1 - 2\Im\mathcal{F}(\mathbf{p})] \\
&\quad - \hbar \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} |f(\mathbf{p})|^2 \partial_{p_i} \Re\mathcal{F}(\mathbf{p}) \\
&\quad - \frac{i}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} |f(\mathbf{p})|^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} \left(\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial}_{p_i} \mathcal{A}^\mu(\mathbf{p},\mathbf{k})\right) \\
&\quad - \frac{i}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} m \left(\frac{f^*(\mathbf{p})u^\dagger_\alpha(P)}{\sqrt{p_0}}\right) \overleftrightarrow{\partial}_{P_i} \left(\frac{f(\mathbf{p})u_\alpha(P)}{\sqrt{p_0}}\right) \mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})\,.
\end{aligned}
\tag{569}
$$

Using the unitarity condition (568) we remove the imaginary part of $\mathcal{F}$ to produce

$$
\begin{aligned}
&\langle x^i(0)\rangle \\
&= i\hbar \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} m \left(\frac{f^*(\mathbf{p})u^\dagger_\alpha(p)}{\sqrt{p_0}}\right) \frac{\partial}{\partial p_i} \left(\frac{f(\mathbf{p})u_\alpha(p)}{\sqrt{p_0}}\right) \\
&\quad - \hbar \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} |f(\mathbf{p})|^2 \partial_p \Re\mathcal{F}(\mathbf{p}) \\
&\quad - \frac{i}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} |f(\mathbf{p})|^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} \mathcal{A}^*_\mu(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial}_{p_i} \mathcal{A}^\mu(\mathbf{p},\mathbf{k}) \\
&\quad + \frac{i}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} m \mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) \\
&\left[\frac{P_0}{p_0}\left(\frac{f^*(\mathbf{p})u^\dagger_\alpha(p)}{\sqrt{p_0}}\right) \overleftrightarrow{\partial}_{p_i} \left(\frac{f(\mathbf{p})u_\alpha(p)}{\sqrt{p_0}}\right) - \left(\frac{f^*(\mathbf{p})u^\dagger_\alpha(P)}{\sqrt{p_0}}\right) \overleftrightarrow{\partial}_{P_i} \left(\frac{f(\mathbf{p})u_\alpha(P)}{\sqrt{p_0}}\right)\right].
\end{aligned}
\tag{570}
$$

To $\mathcal{O}(\hbar^0)$ and using the sharply peaked property of $f(\mathbf{p})$ we thus obtain

$$
\begin{aligned}
\langle x^i(0)\rangle = {}& \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}) \overleftrightarrow{\partial}_{p_i} f(\mathbf{p}) \\
&- \hbar\partial_{p_i}\Re\mathcal{F}(\mathbf{p}) - \frac{i}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0} \mathcal{A}^*_\mu(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial}_{p_i} \mathcal{A}^\mu(\mathbf{p},\mathbf{k})\,.
\end{aligned}
\tag{571}
$$

The first term is the position of the non-radiating particle which we recall is at the origin. There are thus two contributions to the position shift, the emission shift $\delta x^i_{\rm em}$ and forward scattering shift $\delta x^i_{\rm for}$ defined as follows:

$$\delta x^i_{\rm em} = -\,\frac{i}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\ \overleftrightarrow{\partial}_{p_i}\ \mathcal{A}^\mu(\mathbf{p},\mathbf{k}) \tag{572}$$

$$\delta x^i_{\rm for} = -\,\hbar\partial_{p_i}\Re\mathcal{F}(\mathbf{p})\,. \tag{573}$$

In this limit the expression for the position expectation value, in terms of the amplitudes, is the same as that obtained for the scalar field. It is now our task to evaluate these two expressions.

3. Emission Amplitude

The emission process, resulting from the first order interaction term, is given by

$$b^\dagger_\alpha(\mathbf{p})|0\rangle \to \ldots - \frac{i}{\hbar}\int d^4x\,\mathcal{H}_I(x)b^\dagger_\alpha(\mathbf{p})|0\rangle\,. \tag{574}$$

The QED interaction Hamiltonian for the coupling of the spinor and electromagnetic fields is the negative of the interaction Lagrangian.[4] Unlike the more complicated situation we had to deal with for the scalar field, we have just the one coupling term to consider. Substituting the concrete expression for $\mathcal{H}_I$,

[4]As in the scalar case, we note that we are using the free-field normal ordering operators in the interaction Hamiltonian (see footnote 8 at the beginning of the scalar Emission Amplitude calculation). Again, however, it can be shown that this is justified to order $\hbar^2$ [**9**].

and we obtain

$$
\begin{aligned}
& -\frac{i}{\hbar}\int d^4x\mathcal{H}_I(x)b^\dagger_\alpha(\mathbf{p})|0\rangle \\
&= -\frac{ie}{\hbar}\int d^4x\ :A_\mu\bar{\psi}\gamma^\mu\psi:b^\dagger_\alpha(\mathbf{p})|0\rangle \\
&= -\frac{ie}{\hbar}\int d^4x\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3 p'_0/m}\frac{d^3\mathbf{p}''}{(2\pi\hbar)^3 p''_0/m} \\
&\quad\times a^\dagger_\mu(\mathbf{k})e^{ik\cdot x}\bar{\Phi}^\beta(\mathbf{p}')\gamma^\mu\Phi^\gamma(\mathbf{p}'')b^\dagger_\beta(\mathbf{p}')b_\gamma(\mathbf{p}'')b^\dagger_\alpha(\mathbf{p})|0\rangle \\
&= -\frac{ie}{\hbar}\int d^4x\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3 p'_0/m}e^{ik\cdot x}\bar{\Phi}^\beta(\mathbf{p}')\gamma^\mu\Phi_\alpha(\mathbf{p})a^\dagger_\mu(\mathbf{k})b^\dagger_\beta(\mathbf{p}')|0\rangle\,, \quad (575)
\end{aligned}
$$

where we have ignored the separate particle creation vacuum process which is not part of the evolution of the state. The emission amplitude is thus given by the expression

$$
\mathcal{A}^{(\beta)\mu}_{(\alpha)}(\mathbf{p},\mathbf{k}) = -e\int d^4x\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3 p'_0/m}\bar{\Phi}^\beta(\mathbf{p}')\gamma^\mu\Phi_\alpha(\mathbf{p})e^{ik\cdot x}\,. \quad (576)
$$

We have indices on the amplitude to represent the initial and final spins. The fields involved in the interaction are the non-free fields from the region $\mathcal{M}_I$. The mode functions in the emission amplitude are therefore those for the non-free field. We proceed substituting the semiclassical expansion of these mode functions from (202). As we wish to take the $\hbar \to 0$ limit, we shall only need the $\mathcal{O}(\hbar^0)$ terms in the expansion.

$$
\begin{aligned}
\mathcal{A}^{(\beta)\mu}_{(\alpha)}(\mathbf{p},\mathbf{k}) &= -\,e\int\frac{d^4x\,d^3p'}{(2\pi\hbar)^3 p'_0/m}\phi^*_{p'}(t)\phi_p(t)e^{-i\mathbf{p}'\cdot\mathbf{x}/\hbar}e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}e^{ik\cdot x}\bar{u}^\beta(p',t)\gamma^\mu u_\alpha(p,t) \\
&= -\,e\int\frac{d^4x\,d^3p'}{(2\pi\hbar)^3 p'_0/m}\phi^*_{p'}(t)\phi_p(t)e^{ikt}\bar{u}^\beta(p',t)\gamma^\mu u_\alpha(p,t)e^{i(\mathbf{p}-\mathbf{p}'-\hbar\mathbf{k})/\hbar}\,.
\end{aligned}
\quad (577)
$$

The spatial integration gives the delta function corresponding to the conservation of momentum $\mathbf{p} = \mathbf{p}'+\hbar\mathbf{k}$. In our previous working, we stated that we had conservation of momentum and defined the final momentum as $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$. The above calculation demonstrates this conservation (with the substitution

$\mathbf{p}' = \mathbf{P}$).

$$\mathcal{A}^{(\beta)\mu}_{(\alpha)}(\mathbf{p},\mathbf{k}) = -e\int \frac{dt\, d^3\mathbf{p}'}{p_0'/m}\sqrt{\frac{p_0' p_0}{E_{p'}E_p}}\bar{u}^\beta(p',t)\gamma^\mu u_\alpha(p,t) \\ \times \exp\left(-\frac{i}{\hbar}\int_0^t \left(E_p(\zeta) - E_{p'}(\zeta)\right) d\zeta\right) e^{ikt}\delta^3(\mathbf{p} - \mathbf{p}' - \hbar\mathbf{k})\,. \tag{578}$$

The exponential can be written (using the delta function) as in the scalar case:

$$\begin{aligned}\exp\left(-\frac{i}{\hbar}\int_0^t \left(E_p(\zeta) - E_{p'}(\zeta)\right) d\zeta\right) &= \exp\left(-\frac{i}{\hbar}\left[\partial_{\mathbf{p}}\int_0^t E_p(\zeta) d\zeta\right]\cdot[\mathbf{p} - \mathbf{p}']\right) \\ &= \exp\left(-\frac{i}{\hbar}\left[\int_0^t \frac{d\mathbf{x}}{d\zeta}d\zeta\right]\cdot[\hbar\mathbf{k}]\right) \\ &= \exp\left(-i\mathbf{k}\cdot\mathbf{x}\right)\,. \end{aligned}\tag{579}$$

We now look at the spinor factor. The component with γ^0 is to order $\hbar^0$, using the zeroth order spinor given in (178),

$$\begin{aligned}&\bar{u}^\beta(p',t)\gamma^0 u_\alpha(p,t) \\ =&\frac{\sqrt{(E_{p'}+m)(E_p+m)}}{2m}s^{\beta\dagger}U^\dagger_{p'}(t)\left[1 + \frac{\boldsymbol{\sigma}\cdot\mathbf{p}'(t)\,\boldsymbol{\sigma}\cdot\mathbf{p}(t)}{(E_{p'}+m)(E_p+m)}\right]U_p(t)s_\alpha\,,\end{aligned}\tag{580}$$

with $U_p(t)$ defined in (176). We can take $\mathbf{p}'$ to $\mathbf{p}$ in all the terms to lowest $\hbar$ order, including the unitary matrix $U^\dagger_{p'}(t)$. This component thus simplifies to

$$\bar{u}^\beta(p',t)\gamma^0 u^\alpha(p,t) = \frac{E_p}{m}\delta^\beta_\alpha + \mathcal{O}(\hbar)\,. \tag{581}$$

Similarly,

$$\begin{aligned}&\bar{u}^\beta(p',t)\gamma^i u_\alpha(p,t) \\ =&\frac{\sqrt{(E_{p'}+m)(E_p+m)}}{2m}s^{\beta\dagger}U^\dagger_{p'}(t)\left[\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'(t)\,\sigma^i}{E_{p'}+m} + \frac{\sigma^i\,\boldsymbol{\sigma}\cdot\mathbf{p}(t)}{E_p+m}\right]U_p(t)s_\alpha + \mathcal{O}(\hbar)\,.\end{aligned}\tag{582}$$

As before, we change $\mathbf{p}' \to \mathbf{p}$ to the lowest order and note that

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{p}(t)\,\sigma^i}{E_{p'}+m} + \frac{\sigma^i\,\boldsymbol{\sigma}\cdot\mathbf{p}(t)}{E_p+m} = 2\frac{p^i(t)}{E_p+m}\delta^\beta_\alpha\,. \tag{583}$$

Overall, we thus obtain

$$\bar{u}^{\beta}(p',t)\gamma^{\mu}u_{\alpha}(p,t) = \frac{\tilde{p}^{\mu}}{m}\delta^{\beta}_{\alpha} + \mathcal{O}(\hbar)\,. \tag{584}$$

Consequently, the emission amplitude can be written to lowest order as

$$\begin{aligned}\mathcal{A}^{(\beta)\mu}_{(\alpha)}(p,k) &= -\,e\int dt\frac{\tilde{p}^{\mu}}{E_p}e^{ik\cdot x}\delta^{\beta}_{\alpha}\\ &= -\,e\int dt\frac{dx^{\mu}}{dt}e^{ik\cdot x}\delta^{\beta}_{\alpha}\\ &= -\,e\int d\xi\frac{dx^{\mu}}{d\xi}e^{ik\xi}\delta^{\beta}_{\alpha}\,,\end{aligned} \tag{585}$$

where we define $\xi := t - \mathbf{k}\cdot\mathbf{x}/k_0$. We see that as stated previously, the spin does not change in the lowest $\hbar$ order. Therefore

$$\mathcal{A}^{\mu}(p,k) = -\,e\int d\xi\frac{dx^{\mu}}{d\xi}e^{ik\xi}\,. \tag{586}$$

This is the same expression as obtained for the lowest order emission amplitude for the scalar field (344) and is written in terms of the classical trajectory. We additionally note that the amplitude is equal to the classical amplitude. Consequently, using either of the methods from the scalar calculations (Chapter 4 section 3 and Chapter 5 section 1) we find that the position shift due to emission δx^i_{em} is equal to the classical position shift δx^i_C. From Chapter 5 we can rewrite the shift as follows:

$$\begin{aligned}\delta x^i_{\mathrm{em}} &= -\frac{i}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\left(\mathcal{A}^*_{\mu}(\mathbf{p},\mathbf{k})\partial_{p_i}\mathcal{A}^{\mu}(\mathbf{p},\mathbf{k})\right)\\ &= \int d^4x\,\partial_{p^i}j^{\mu}(x)A_{\mathrm{R}\mu}(x)\\ &= -\int_{-\infty}^{0}dt\,\mathcal{F}^j_{\mathrm{LD}}\left(\frac{\partial x^j}{\partial p^i}\right)_t\,,\end{aligned} \tag{587}$$

where $A_{\mathrm{R}\mu}(x)$ is the regular field, constructed from the regular Green's function $1/2(G_- - G_+)$, which acts on the classical particle to produce the radiation reaction force [**18**]. This was the same situation we had for the scalar field, whereby the position shift due to the emission could be equated with that due to the regular (or radiative) field and consequently the full classical position

shift. As with the previous case though, we still have another quantum contribution to the position shift from the forward scattering which we must take into account and calculate.

4. Forward Scattering

We now consider the forward scattering amplitude. As before, we calculate the amplitude using the semiclassical mode functions for the interacting region $\mathcal{M}_I$ and expect this result to also be divergent. To this we can add the amplitude due to the QED mass counterterm, thereby renormalising the forward scattering. It is well known that the counter term is also divergent. We shall see however, that the situation is not as straightforward as the scalar renormalisation. Before continuing with the amplitude calculation, we briefly recall that the position shift term is

$$\delta x^i_{\text{for}} = -\hbar\partial_{p_i}\Re\mathcal{F}(\mathbf{p})\,, \tag{573}$$

containing a multiplying factor of $\hbar$. Given that the forward scattering is of order $\hbar^{-2}$, we are interested in the $\hbar^{-2}$ and $\hbar^{-1}$ terms of the real part of $\mathcal{F}$ leading to position shift contributions at order $\hbar^{-1}$ and $\hbar^{0}$ respectively. For the scalar field, the former contribution canceled upon renormalisation, whereas the latter contribution was zero due to $\mathcal{F}$ being imaginary at that order. With these comments and the previous method and calculation in mind, we proceed with the spinor amplitude.

The relative simplicity of the interaction Hamiltonian for the spinor field when compared with the previous scalar case means that the forward scattering process is simply the one-loop process and is the zero-photon sector of the second order interaction term:

$$i\mathcal{F}(\mathbf{p})b^\dagger_\alpha(\mathbf{p})|0\rangle = \frac{1}{2}\left(\frac{-i}{\hbar}\right)^2\int d^4x d^4x' T\left[\mathcal{H}_I(x')\mathcal{H}_I(x)\right]b^\dagger_\alpha(\mathbf{p})|0\rangle\bigg|_{\text{zero}-\text{photon}}\,, \tag{588}$$

with $\mathcal{H}_I = e\bar{\psi}\not{A}\psi$. Operating on both sides with $\langle 0|b_\alpha(\mathbf{p}')$ we have

$$\langle 0|b_\alpha(\mathbf{p}')i\mathcal{F}(\mathbf{p})b^\dagger_\alpha(\mathbf{p})|0\rangle$$
$$= -\frac{1}{2}\frac{1}{\hbar^2}\int d^4x d^4x' \langle 0|b_\alpha(\mathbf{p}')T\left[: e\bar{\psi}(x')\not{A}(x')\psi(x') :: e\bar{\psi}(x)\not{A}(x)\psi(x) :\right] b^\dagger_\alpha(\mathbf{p})|0\rangle \,,$$

thus

$$\mathcal{F}(\mathbf{p}) = \frac{i}{2}\frac{e^2 m}{\hbar^2 p_0}\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3} d^4x d^4x' \langle 0|T\left[A_\mu(x')A_\nu(x)\right]|0\rangle$$
$$\times \langle 0|b_\alpha(\mathbf{p}')T\left[: \bar{\psi}(x')\gamma^\mu\psi(x') :: \bar{\psi}(x)\gamma^\nu\psi(x) :\right] b^\dagger_\alpha(\mathbf{p})|0\rangle \,. \quad (589)$$

Here we have, as one would expect from a one-loop diagram, the photon propagator

$$\langle 0|T\left[A_\mu(x')A_\nu(x)\right]|0\rangle$$
$$= -\hbar g_{\mu\nu}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\left[\theta(t'-t)e^{-ik\cdot(x'-x)} + \theta(t-t')e^{-ik\cdot(x-x')}\right] \,, \quad (590)$$

and a time-ordered combination of the spinor fields which we shall denote

$$T_S(x,x') = \langle 0|b_\alpha(\mathbf{p}')T\left[: \bar{\psi}(x')\gamma^\mu\psi(x') :: \bar{\psi}(x)\gamma^\nu\psi(x) :\right] b^\dagger_\alpha(\mathbf{p})|0\rangle \,. \quad (591)$$

Writing T_S with the field expansions the term for each normal ordered product, in which we have only one space-time variable, is of the form

$$: \bar{\psi}\gamma^\nu\psi := \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{m}{p_0}\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{m}{p_0'}$$
$$: \left[b^\dagger_\alpha(\mathbf{p})\bar{\Phi}^\alpha(\mathbf{p}) + d_\alpha(\mathbf{p})\bar{\Psi}^\alpha(\mathbf{p})\right]\gamma^\nu\left[b_\beta(\mathbf{p}')\Phi^\beta(\mathbf{p}') + d^\dagger_\beta(\mathbf{p}')\Psi^\beta(\mathbf{p}')\right] : \,. \quad (592)$$

The creation and annihilation operators form the following combinations

$$: b^\dagger_\alpha(\mathbf{p})b_\beta(\mathbf{p}') + d_\alpha(\mathbf{p})b_\beta(\mathbf{p}') + b^\dagger_\alpha(\mathbf{p})d^\dagger_\beta(\mathbf{p}') + d_\alpha(\mathbf{p})d^\dagger_\beta(\mathbf{p}') :$$
$$= b^\dagger_\alpha(\mathbf{p})b_\beta(\mathbf{p}') + d_\alpha(\mathbf{p})b_\beta(\mathbf{p}') + b^\dagger_\alpha(\mathbf{p})d^\dagger_\beta(\mathbf{p}') - d^\dagger_\beta(\mathbf{p}')d_\alpha(\mathbf{p}) \,. \quad (593)$$

When operated on the left by $\langle 0|b_\gamma(\mathbf{p}'')$ the third and fourth terms vanish and when operated on the right by $b^\dagger_\gamma(\mathbf{p}'')|0\rangle$, the second and fourth terms drop

out. Hence, in terms of the operators only, the form of T_S is

$$\begin{aligned}&\langle 0|b_\alpha(\mathbf{p}')\left(b^\dagger_\beta b_\gamma + d_\beta b_\gamma\right)\left(b^\dagger_\delta b_\epsilon + b^\dagger_\delta d^\dagger_\epsilon\right) b^\dagger_\alpha(\mathbf{p})|0\rangle \\ &= \langle 0|b_\alpha(\mathbf{p}')b^\dagger_\beta b_\gamma b^\dagger_\delta b_\epsilon b^\dagger_\alpha(\mathbf{p})|0\rangle - \langle 0|b_\alpha(\mathbf{p}')b^\dagger_\delta d_\beta d^\dagger_\epsilon b_\gamma b^\dagger_\alpha(\mathbf{p})|0\rangle + V\,, \end{aligned} \tag{594}$$

where V is the process with a vacuum graph (creation of electron-positron pair and photon and subsequent annihilation separate from the original particle and an unaffected particle) which as before, we can ignore. In both the above calculations, we have the familiar minus signs due to the anticommutation relations used for spinor fields. Reintroducing the mode functions is straightforward by matching their spins indices to those on the creation/annihilation operators in the expansion above:

$$\begin{aligned} T_S(x,x') = \int & \frac{d^3\mathbf{p^{(2)}}m}{(2\pi\hbar)^3 p_0^{(2)}}\frac{d^3\mathbf{p^{(3)}}m}{(2\pi\hbar)^3 p_0^{(3)}}\frac{d^3\mathbf{p^{(4)}}m}{(2\pi\hbar)^3 p_0^{(4)}}\frac{d^3\mathbf{p^{(5)}}m}{(2\pi\hbar)^3 p_0^{(5)}} \\ & T\Big[\bar{\Phi}^\beta(\mathbf{p^{(2)}},x)\gamma^\mu\Phi^\gamma(\mathbf{p^{(3)}},x)\bar{\Phi}^\delta(\mathbf{p^{(4)}},x')\gamma^\nu\Phi^\epsilon(\mathbf{p^{(5)}},x') \\ & \quad\times \langle 0|b_\alpha(\mathbf{p}')b^\dagger_\beta(\mathbf{p^{(2)}})b_\gamma(\mathbf{p^{(3)}})b^\dagger_\delta(\mathbf{p^{(4)}})b_\epsilon(\mathbf{p^{(5)}})b^\dagger_\alpha(\mathbf{p})|0\rangle \\ & - \bar{\Psi}^\beta(\mathbf{p^{(2)}},x)\gamma^\mu\Phi^\gamma(\mathbf{p^{(3)}},x)\bar{\Phi}^\delta(\mathbf{p^{(4)}},x')\gamma^\nu\Psi^\epsilon(\mathbf{p^{(5)}},x') \\ & \quad\times \langle 0|b_\alpha(\mathbf{p}')b^\dagger_\delta(\mathbf{p^{(4)}})d_\beta(\mathbf{p^{(2)}})d^\dagger_\epsilon(\mathbf{p^{(5)}})b_\gamma(\mathbf{p^{(3)}})b^\dagger_\alpha(\mathbf{p})|0\rangle\Big]\,, \end{aligned} \tag{595}$$

which simplifies when we use the anticommutation relations, to produce

$$\begin{aligned} T_S(x,x') = \int \frac{d^3\mathbf{q}m}{(2\pi\hbar)^3 q_0} T\, & \big[\bar{\Phi}_\alpha(\mathbf{p}',x)\gamma^\mu\Phi^\gamma(\mathbf{q},x)\bar{\Phi}_\gamma(\mathbf{q},x')\gamma^\nu\Phi_\alpha(\mathbf{p},x') \\ & -\bar{\Psi}^\beta(\mathbf{q},x)\gamma^\mu\Phi_\alpha(\mathbf{p},x)\bar{\Phi}_\alpha(\mathbf{p}',x')\gamma^\nu\Psi_\beta(\mathbf{q},x')\big]\,, \end{aligned} \tag{596}$$

where we have changed the remaining integration variable to $\mathbf{q}$ in both cases. The reader will recall that in the forward scattering (589), the product T_S is integrated over x and x' and contracted with the metric $g_{\mu\nu}$. As a consequence of the resulting symmetry, both orders of the time variables give the same result and therefore we take one and multiply by two. The forward scattering (not

forgetting the photon propagator term) can then be written

$$\mathcal{F}(\mathbf{p}) = -\frac{ie^2m}{\hbar p_0}\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{d^3\mathbf{q}m}{(2\pi\hbar)^3q_0}d^4xd^4x'\theta(t-t')e^{-ik\cdot(x-x')}$$
$$\Big[\bar{\Phi}_\alpha(\mathbf{p}',x)\gamma^\mu\Phi^\gamma(\mathbf{q},x)\bar{\Phi}_\gamma(\mathbf{q},x')\gamma_\mu\Phi_\alpha(\mathbf{p},x')$$
$$-\bar{\Psi}^\beta(\mathbf{q},x)\gamma^\mu\Phi_\alpha(\mathbf{p},x)\bar{\Phi}_\alpha(\mathbf{p}',x')\gamma_\mu\Psi_\beta(\mathbf{q},x')\Big] \ . \quad (597)$$

As with the one-loop part of the scalar field, we split this up into the 'particle loop' and 'anti-particle loop', defined respectively by[5]

$$\mathcal{F}_-(\mathbf{p}) = -\frac{ie^2m}{\hbar p_0}\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{d^3\mathbf{q}m}{(2\pi\hbar)^3q_0}d^4xd^4x'\theta(t-t')$$
$$\times\,\bar{\Phi}_\alpha(\mathbf{p}',x)\gamma^\mu\Phi^\gamma(\mathbf{q},x)\bar{\Phi}_\gamma(\mathbf{q},x')\gamma_\mu\Phi_\alpha(\mathbf{p},x')e^{-ik\cdot(x-x')}\,, \quad (598)$$

$$\mathcal{F}_+(\mathbf{p}) = \frac{ie^2m}{\hbar p_0}\int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{d^3\mathbf{q}m}{(2\pi\hbar)^3q_0}d^4xd^4x'\theta(t-t')$$
$$\times\,\bar{\Phi}_\alpha(\mathbf{p}',x')\gamma^\mu\Psi_\beta(\mathbf{q},x')\bar{\Psi}^\beta(\mathbf{q},x)\gamma_\mu\Phi_\alpha(\mathbf{p},x)e^{-ik\cdot(x-x')}\,, \quad (599)$$

where we have rearranged the spinor mode functions in the last line.

4.1. Particle Loop. The $\mathcal{F}_-(\mathbf{p})$ half of the forward scattering amplitude is the result of the particle loop process. Using the semiclassical expansion, the spinor mode functions in $\mathcal{F}_-(\mathbf{p})$, along with the k exponential,[6] can be written

$$\bar{\Phi}_\alpha(\mathbf{p}',x)\gamma^\mu\Phi^\gamma(\mathbf{q},x)\bar{\Phi}_\gamma(\mathbf{q},x')\gamma_\mu\Phi_\alpha(\mathbf{p},x')e^{-ik\cdot(x-x')}$$
$$= [\bar{u}_\alpha(p',t)\gamma^\mu u^\gamma(q,t)\bar{u}_\gamma(q,t')\gamma_\mu u_\alpha(p,t')]$$
$$\times\,\phi^*_{p'}(t)\phi_q(t)\phi^*_q(t')\phi_p(t')e^{-i\mathbf{p}'\cdot\mathbf{x}/\hbar}e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}e^{-i\mathbf{q}\cdot\mathbf{x}'/\hbar}e^{i\mathbf{p}\cdot\mathbf{x}'/\hbar}e^{-ik\cdot(x-x')}\,, \quad (600)$$

where we have the time-dependent spinor expansions inside the square brackets and the scalar semiclassical terms and the exponentials outside. Completing the spatial integrals produces the delta functions $\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{K})\delta^3(\mathbf{p}'-\mathbf{q}-\mathbf{K})$

[5]The plus and minus designation will become clear later. It is however the same as that used in the scalar $\mathcal{F}_2$ cases.

[6]which is really the free electromagnetic field mode function,

with $\mathbf{K} := \hbar\mathbf{k}$. Using $\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{K})\delta^3(\mathbf{p}-\mathbf{p}')$ and integrating over $\mathbf{p}'$, the particle loop contribution can be written

$$\begin{aligned}\mathcal{F}_-(\mathbf{p}) = -\frac{ie^2m}{\hbar p_0}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{d^3\mathbf{q}m}{(2\pi\hbar)^3 q_0} dtdt'\theta(t-t')(2\pi\hbar)^3\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{K}) \\ \times\left[\mathcal{S}_-\right]\phi_p^*(t)\phi_q(t)\phi_q^*(t')\phi_p(t')e^{-iK(t-t')/\hbar}\,, \end{aligned} \tag{601}$$

where we have defined the semiclassical particle loop spinor combination

$$\mathcal{S}_- = \bar{u}_\alpha(p,t)\gamma^\mu u^\gamma(q,t)\bar{u}_\gamma(q,t')\gamma_\mu u_\alpha(p,t')\,. \tag{602}$$

We shall at times refer to the spinors in between the γ matrices as the *inner* spinors, and those on the outside as the *outer* spinors. Following our previous method, the next step is to change the time variables of integration in order to expand the difference in terms of $\hbar$, i.e.

$$t = \bar{t} - \frac{\hbar}{2}\eta \tag{603}$$

$$t' = \bar{t} + \frac{\hbar}{2}\eta\,, \tag{604}$$

with $\theta(t-t') = \theta(-\eta)$ and

$$dtdt' = \hbar dt d\eta\,. \tag{605}$$

The semiclassical scalar component and the action of this variable change is familiar from the scalar field calculations and we present a summary here. We have

$$\phi_{\mathbf{p}}(t) = \sqrt{\frac{p_0}{E_p(t)}}\exp\left[-\frac{i}{\hbar}\int_0^t E_p(t')\,dt'\right]\,. \tag{606}$$

The term $\varphi_{\mathbf{p}}(t)$, which in (443) represented the higher order expansion terms, is not used here as these terms are contained within the expansion of the spinors. The product can be written (mixing both variable sets) by

$$\phi_p^*(t)\phi_p(t') = \frac{p_0}{\sqrt{E_p(t)E_p(t')}}\exp\left[-i\int_{-\eta/2}^{\eta/2} E_p(\bar{t}+\hbar\zeta)\,d\zeta\right]\,. \tag{607}$$

Changing the variables in the prefactor to the exponential, the product is equal to $|\phi_p(\bar{t})|^2 + \mathcal{O}(\hbar^2)$, which leads us overall to produce

$$\begin{aligned}&\phi_p^*(t)\phi_q(t)\phi_q^*(t')\phi_p(t')e^{iK(t-t')/\hbar}\\&= |\phi_p(\bar{t})|^2|\phi_q(\bar{t})|^2 \exp\left[i\int_{-\eta/2}^{\eta/2}\left(-E_p(\bar{t}+\hbar\zeta)+E_q(\bar{t}+\hbar\zeta)+K\right)\,d\zeta\right]. \quad (608)\end{aligned}$$

We recall that we shall need to be careful when integrating over η due to the infrared divergence. As before we can use the variable $\tilde{\beta}$ to replace η and aid the integration, where

$$[-E_p(t)+E_q(t)+K]\tilde{\beta} \equiv \int_{-\eta/2}^{\eta/2}\left[-E_p(t+\hbar\zeta)+E_q(t+\hbar\zeta)+K\right]\,d\zeta\,. \quad (609)$$

We have already studied and confirmed the validity of this transformation when dealing with the scalar field, and the same argument can be repeated here briefly to aid recall. For low $K = \|\mathbf{p}-\mathbf{q}\| \to 0$, we have the same equation as before:

$$-E_p(t)+E_q(t)+K \approx K - \mathbf{v}(t)\cdot\mathbf{K}\,, \quad (457)$$

where $\mathbf{v}(t) = [\mathbf{p}-\mathbf{V}(t)]/E_p(t)$ is the velocity of the classical particle with final momentum $\mathbf{p}$ and thus with $\mathbf{n} \equiv \mathbf{K}/K$,

$$\tilde{\beta} = \frac{1}{1-\mathbf{v}(t)\cdot\mathbf{n}}\int_{-\eta/2}^{\eta/2}\left[1-\mathbf{v}(t+\hbar\zeta)\cdot\mathbf{n}\right]d\zeta\,. \quad (610)$$

Therefore, writing $d\eta = J(\mathbf{p},\mathbf{q},t,\hbar\tilde{\beta})d\tilde{\beta}$, the function $J(\mathbf{p},\mathbf{q},t,\hbar\tilde{\beta})$ is finite as $K \to 0$ and we can use the variable $\tilde{\beta}$ to replace η. The expansions of η and $d\eta$ are

$$\eta = \left[1 - \frac{1}{24}\frac{\left(-\ddot{E}_{\mathbf{p}}(t)+\ddot{E}_{\mathbf{q}}(t)\right)}{(-E_p(t)+E_q(t)+K)}\hbar^2\tilde{\beta}^2 + \mathcal{O}(\hbar^4\tilde{\beta}^4)\right]\tilde{\beta}\,, \quad (611)$$

and

$$d\eta = \left[1 - \frac{1}{8}\frac{\left(-\ddot{E}_{\mathbf{p}}(t)+\ddot{E}_{\mathbf{q}}(t)\right)}{(-E_p(t)+E_q(t)+K)}\hbar^2\tilde{\beta}^2 + \mathcal{O}(\hbar^4\tilde{\beta}^4)\right]d\tilde{\beta}\,. \quad (612)$$

Integration of $\tilde{\beta}$ will produce further powers of $(-E_p(t)+E_q(t)+K)$ in the denominator, thus producing an infrared divergences as $\lim_{K\to 0}[-E_p(t)+E_q(t)+$

$K] \to 0$ as before. Before completing this integration, we must return to the spinor combination $\mathcal{S}_-$.

The particle loop spinor combination, we recall, was defined as

$$\mathcal{S}_- = \bar{u}_\alpha(p,t)\gamma^\mu u_\beta(q,t)\bar{u}^\beta(q,t')\gamma_\mu u_\alpha(p,t')\,. \tag{613}$$

The spinors here are the time-dependent semiclassical expansion spinors derived previously Chapter 2, section 3. The relevant identities and $\hbar$ expansions, including the change of time variables to $(\bar{t},\eta)$, to enable the calculation of $\mathcal{S}_-$ are detailed in the Appendix A, with a summary in section 6. We quote the necessary results as they are needed here. Building up the $\mathcal{S}_-$ combination from the inner spinors, we have[7] to $\mathcal{O}(\hbar)$

$$\begin{aligned} &u_\alpha(q,t)\bar{u}^\alpha(q,t') \\ &= \frac{\gamma\cdot\tilde{q}+m}{2m} + \left(\frac{\hbar}{m(2E_q)^2} - \frac{i\hbar\eta}{2m2E_q}\right)\gamma^5\boldsymbol{\gamma}\cdot\tilde{\mathbf{q}}\times\dot{\tilde{\mathbf{q}}} - \left(\frac{i\hbar}{(2E_q)^2} + \frac{\hbar\eta}{2E_q.2}\right)\gamma^0\boldsymbol{\gamma}\cdot\dot{\tilde{\mathbf{q}}}\,. \end{aligned} \tag{614}$$

Here and later all energies and momenta without explicit time arguments are evaluated at $\bar{t}$. Sandwiching the expression immediately above between the contracted gamma matrices, we find

$$\gamma^\mu u_\alpha(q,t)\bar{u}^\alpha(q,t')\gamma_\mu = 2 - \frac{\gamma\cdot\tilde{q}}{m} + \left(\frac{\hbar}{2mE_q^2} - \frac{i\hbar\eta}{2mE_q}\right)\gamma^5\boldsymbol{\gamma}\cdot\tilde{\mathbf{q}}\times\dot{\tilde{\mathbf{q}}}\,. \tag{615}$$

The particle loop combination can then be written

$$\begin{aligned} \mathcal{S}_- &= \bar{u}_\alpha(p,t)\gamma^\mu u_\beta(q,t)\bar{u}^\beta(q,t')\gamma_\mu u_\alpha(p,t') \\ &= 2\left[\bar{u}_\alpha(p,t)u_\alpha(p,t')\right] - \frac{E_q}{m}\left[\bar{u}_\alpha(p,t)\gamma^0 u_\alpha(p,t')\right] + \frac{\tilde{\mathbf{q}}}{m}\cdot\left[\bar{u}_\alpha(p,t)\boldsymbol{\gamma}u_\alpha(p,t')\right] \\ &\quad + \left(\frac{\hbar}{2mE_q^2} - \frac{i\hbar\eta}{2mE_q}\right)\left[\bar{u}_\alpha(p,t)\gamma^5\boldsymbol{\gamma}\cdot\tilde{\mathbf{q}}\times\dot{\tilde{\mathbf{q}}}u_\alpha(p,t')\right]\,. \end{aligned} \tag{616}$$

[7] Recall that $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{V}(t)$ and similarly for $\tilde{\mathbf{p}}$.

Substituting into this expression the relevant spinor identities, we have

$$\begin{aligned}
\mathcal{S}_- &= 2\left[1-\left(\frac{\hbar}{2mE_p}+\frac{i\hbar\eta}{2m}\right)\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p}\right]-\frac{E_q}{m}\left[\frac{E_p}{m}\right] \\
&\quad+\frac{\tilde{\mathbf{q}}}{m}\cdot\left[\frac{\tilde{\mathbf{p}}}{m}-\left(\frac{\hbar}{2mE_p}+\frac{i\hbar\eta}{2m}\right)\left(\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\,\tilde{\mathbf{p}}}{E_p(E_p+m)}+\boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}}-\frac{\dot{E}_p\tilde{\mathbf{p}}}{E_p+m}\right)\right)\right] \\
&\quad-\left(\frac{\hbar}{2mE_q^2}-\frac{i\hbar\eta}{2mE_q}\right)\left[\boldsymbol{\xi}\cdot\tilde{\mathbf{q}}\times\dot{\tilde{\mathbf{q}}}+\frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{q}}\times\dot{\tilde{\mathbf{q}}}\,\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}}{m(E_p+m)}\right] \\
&= 2-\frac{\tilde{q}\cdot\tilde{p}}{m^2}-\left(\frac{\hbar}{mE_p}+\frac{i\hbar\eta}{m}\right)\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p} \\
&\quad-\left(\frac{\hbar}{2m^2E_p}+\frac{i\hbar\eta}{2m^2}\right)\left(\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\,\tilde{\mathbf{p}}\cdot\tilde{\mathbf{q}}}{E_p(E_p+m)}+\tilde{\mathbf{q}}\cdot\boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}}-\frac{\dot{E}_p\tilde{\mathbf{p}}}{E_p+m}\right)\right) \\
&\quad-\left(\frac{\hbar}{2mE_q^2}-\frac{i\hbar\eta}{2mE_q}\right)\left[\boldsymbol{\xi}\cdot\tilde{\mathbf{q}}\times\dot{\tilde{\mathbf{q}}}+\frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{q}}\times\dot{\tilde{\mathbf{q}}}\,\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}}{m(E_p+m)}\right]. \qquad (617)
\end{aligned}$$

Changing the variables[8] and adding the scalar part of the expansion, we can write the contribution to the forward scattering as

$$\begin{aligned}
\mathcal{F}_-(\mathbf{p}) &= -\frac{ie^2m}{\hbar^2p_0}\int\frac{d^3\mathbf{K}}{2K(2\pi)^3}\frac{d^3\mathbf{q}m}{(2\pi\hbar)^3q_0}d\bar{t}d\eta\theta(-\eta)(2\pi\hbar)^3\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{K}) \\
&\quad\times[\mathcal{S}_-]\,|\phi_p(\bar{t})|^2|\phi_q(\bar{t})|^2\exp\left[i\int_{-\eta/2}^{\eta/2}(-E_p(\bar{t}+\hbar\zeta)+E_q(\bar{t}+\hbar\zeta)+K)\ d\zeta\right]. \qquad (618)
\end{aligned}$$

The factor $\mathcal{S}_-$ is present in the integrand subject to the delta function $\delta(\mathbf{p}-\mathbf{q}-\mathbf{K})$ and the integrals over $d^3\mathbf{K}$ and $d^3\mathbf{q}$. Integrating out this delta function using the $\mathbf{q}$ integral, we can replace $\tilde{\mathbf{q}}$ by $\tilde{\mathbf{q}}\to\tilde{\mathbf{p}}-\mathbf{K}$. We also have $\dot{\tilde{\mathbf{q}}}=\dot{\tilde{\mathbf{p}}}$, as both are in fact equal to $-\dot{\mathbf{V}}$. The energy E_q can then be regarded as defined by $E_q=\sqrt{(\tilde{\mathbf{p}}-\mathbf{K})^2+m^2}$. Within the resulting K integral, the only preferred direction about which to choose an axis is that given by $\tilde{\mathbf{p}}$. We may thus replace $\mathbf{K}$ by $\mathbf{K}\to(\mathbf{K}\cdot\tilde{\mathbf{p}}/\tilde{\mathbf{p}}^2)\,\tilde{\mathbf{p}}$. Combining all of the above statements, we may produce the 'effective version'[9] of $\mathcal{S}_-$ under the integrations via the

[8] $(t,t')\to(\bar{t},\eta)$ variables and also changing to $\mathbf{K}=\hbar\mathbf{k}$,

[9] The effective version under the delta function.

transformation

$$\tilde{\mathbf{q}} \to \tilde{\mathbf{p}}\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) . \tag{619}$$

This produces

$$\begin{aligned}
\mathcal{S}_-^{\text{eff}} &= 2 - \frac{E_q E_p}{m^2} + \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{p}}}{m^2}\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) - \left(\frac{\hbar}{mE_p} + \frac{i\hbar\eta}{m}\right)\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p} \\
&\quad - \left(\frac{\hbar}{2m^2E_p} + \frac{i\hbar\eta}{2m^2}\right)\left(\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\,\tilde{\mathbf{p}}^2}{E_p(E_p+m)} + \tilde{\mathbf{p}}\cdot\boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}} - \frac{\dot{E}_p\tilde{\mathbf{p}}}{E_p+m}\right)\right)\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) \\
&\quad - \left(\frac{\hbar}{2mE_q^2} - \frac{i\hbar\eta}{2mE_q}\right)\left[\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}} + \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{q}}}\,\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}}{m(E_p+m)}\right]\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) \\
&= 2 - \frac{E_q E_p}{m^2} + \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{p}}}{m^2}\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) - \left(\frac{\hbar}{mE_p} + \frac{i\hbar\eta}{m}\right)\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p} \\
&\quad - \left(\frac{\hbar}{2m^2E_p} + \frac{i\hbar\eta}{2m^2}\right)\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left(\frac{-m}{E_p}\right)\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) \\
&\quad - \left(\frac{\hbar}{2mE_q^2} - \frac{i\hbar\eta}{2mE_q}\right)\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) .
\end{aligned} \tag{620}$$

Hence

$$\begin{aligned}
\mathcal{S}_-^{\text{eff}} &= 2 - \frac{E_q E_p}{m^2} + \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{p}}}{m^2} - \frac{\hbar}{2m}\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left(\frac{1}{E_p^2} + \frac{1}{E_q^2}\right) \\
&\quad - \frac{i\hbar\eta}{2m}\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left(\frac{1}{E_p} - \frac{1}{E_q}\right) \\
&\quad - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\left[\frac{\tilde{\mathbf{p}}^2}{m^2} + \boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left(\frac{\hbar}{2m}\left(\frac{1}{E_p^2} - \frac{1}{E_q^2}\right) + \frac{i\hbar\eta}{2m}\left(\frac{1}{E_p} + \frac{1}{E_q}\right)\right)\right] .
\end{aligned} \tag{621}$$

We change variables to $\tilde{\beta}$ instead of η and rewrite the notation $\bar{t}$ as t. Writing $\mathcal{S}_-^{\text{eff}}(t,\tilde{\beta})$ to indicate the new time variable and notation, we now find the forward scattering contribution as

$$\begin{aligned}
\mathcal{F}_-(\mathbf{p}) = -\frac{ie^2m^2}{\hbar^2 p_0}\int \frac{d^3\mathbf{K}}{(2\pi)^3 2Kq_0} dt d\tilde{\beta} J(\mathbf{p},\mathbf{q},t,\hbar\tilde{\beta})\theta(-\tilde{\beta})|\phi_p(t)|^2|\phi_q(t)|^2 \\
\times \mathcal{S}_-^{\text{eff}}(t,\tilde{\beta})\exp\left[i\left(-E_p + E_q + K\right)\tilde{\beta}\right] .
\end{aligned} \tag{622}$$

Within the integrand we have terms of order $\hbar^0$, $\hbar$ and higher order terms resulting from the $\hbar$-expansion of $\eta \to \tilde{\beta}$ in $\mathcal{S}_-^{\text{eff}}(t,\tilde{\beta})$ and in $J(\mathbf{p},\mathbf{q},t,\hbar\tilde{\beta})$. Here

we again note that the higher order terms in $\hbar$ ($\hbar^2$ and above) can no longer be assumed to be zero in the classical limit due to the infrared divergence. Now, note that every occurrence of $\tilde{\beta}$ is of the form $\hbar\tilde{\beta}$. Therefore all higher order terms are of the form $\hbar^n\tilde{\beta}^d f_{nd}$ where f_{nd} is some function (for each choice of n and d) and $n \geq 1$, $n \geq d \geq 0$. To complete the integration over $\tilde{\beta}$ we must use a Wick rotation i.e. replace K by $K - i\epsilon$. We shall then use the integral

$$\lim_{\epsilon\to 0}\int_{-\infty}^{0}\tilde{\beta}^d e^{iX\tilde{\beta}}e^{\epsilon\tilde{\beta}}d\tilde{\beta} = \frac{d!(i)^{d-1}}{X^{d+1}}\,, \tag{623}$$

to integrate over $\tilde{\beta}$. Further noting that we can write $|\phi_q(t)|^2 = q_0/E_q + \mathcal{O}(\hbar^2)$, the result can be written as

$$\mathcal{F}_-(\mathbf{p}) = -\frac{ie^2m^2}{\hbar^2 p_0}\int\frac{d^3\mathbf{K}}{(2\pi)^3 2KE_q}dt|\phi_p(t)|^2$$
$$\times\left[\frac{(-i)\left(f_{00}+\hbar f_{10}\right)}{(-E_p+E_q+K)} + \frac{\hbar f_{11}}{(-E_p+E_q+K)^2} + \sum_{\substack{n\geq 2\\ n\geq d\geq 0}}\frac{\hbar^n f_{nd} d! i^{d-1}}{(-E_p(t)+E_q(t)+K)^{d+1}}\right], \tag{624}$$

where we have written the first three terms explicitly, outside of the summation sign $\sum$. From $\mathcal{S}_-^{\text{eff}}$ we can give these three terms as

$$f_{00} = 2 - \frac{E_qE_p}{m^2} + \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{p}}}{m^2}\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right)\,, \tag{625}$$

$$f_{10} = \frac{1}{2m}\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left[-\left(\frac{1}{E_p^2}+\frac{1}{E_q^2}\right) - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\left(\frac{1}{E_p^2}-\frac{1}{E_q^2}\right)\right]\,, \tag{626}$$

$$f_{11} = \frac{i}{2m}\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left[-\left(\frac{1}{E_p}-\frac{1}{E_q}\right) - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\left(\frac{1}{E_p}+\frac{1}{E_q}\right)\right]\,. \tag{627}$$

Here we can see the infrared divergences in the higher order terms. The remaining terms of the summation, i.e. with $n \geq 2$, have the same form as the higher order terms dealt with in the scalar field calculations. Thus we introduce a cut-off and integrate above $K_0 = \hbar^\alpha\lambda$ with $3/4 < \alpha < 1$ (for later

reasons) and λ a positive constant. The small-K contributions behave like

$$\begin{cases} \hbar^n K_0^{1-d} = \hbar^{n+(1-d)\alpha}\lambda^{1-d} & \text{for } d \geq 2 \\ \hbar \log(\hbar\alpha) & \text{for the } d = 1 \text{ term} \end{cases} . \tag{628}$$

Given the limits on n, d and α, we again have $n+(1-d)\alpha \geq 2-\alpha$. Considering the additional $\hbar^{-2}$ multiplying the integral (after converting k to K) and the $\hbar$ multiplying the whole forward scattering, we find that these higher order terms do not contribute in the $\hbar \to 0$ limit to the position shift. Above the cut-off this leaves the lowest order term and the first order correction to be considered. We must still analyse the contributions below the cut-off, however. Fortunately, this calculation is again analogous to what we previously encountered. We can in fact show that the real contribution to $\mathcal{F}_-$ below that cut-off comes entirely from the leading order term to order $\hbar^{-1}$ [10], which in our current notation is the f_{00} term in (624). The remaining contribution to $\mathcal{F}_-$ at this order is again imaginary. We demonstrate this as previously, by calculating the low-K contribution of the full $\mathcal{F}_-$ and comparing it with the leading order contribution. We additionally recall that it is in the classical limits with which all our results are phrased.

Firstly, below the cut-off $\mathcal{F}_-$ can be written

$$\mathcal{F}_-^< = -\frac{ie^2}{\hbar}\int dt dt' \int_{k<\hbar^{\alpha-1}\lambda} \frac{d^3k}{2k(2\pi)^3}\frac{d^3q}{(2\pi\hbar)^3}\frac{m^2}{q_0 p_0}\theta(t-t')(2\pi\hbar)^3\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{K})$$
$$\times \left[\bar{u}_\alpha(p,t)\gamma^\mu u^\gamma(q,t)\bar{u}_\gamma(q,t')\gamma_\mu u_\alpha(p,t')\right]\phi_p^*(t)\phi_q(t)\phi_q^*(t')\phi_p(t')e^{-K(t-t')/\hbar} . \tag{629}$$

[10] That is order $\hbar^{-1}$ for $\mathcal{F}_-$ which one recalls has a prefactor of $1/\hbar^2$ multiplying the integrand and is additionally multiplied by a further $\hbar$ when producing the position shift contribution.

For small k we have

$$
\begin{aligned}
&\exp\left(i\int_0^t\left(K+E_q(\zeta)-E_p(\zeta)\right)d\zeta/\hbar\right)\\
=&\exp\left(i\int_0^t\left(k-\frac{\partial E_p(\zeta)}{\partial \mathbf{p}}\cdot\mathbf{k}+\frac{1}{2}\frac{\partial^2 E_p(\zeta)}{\partial p^i\partial p^j}\hbar k^i k^j+\dots\right)d\zeta\right)\\
=&\exp\left(ikt-\int_0^t\frac{d\mathbf{x}}{d\zeta}\cdot\mathbf{k}d\zeta\right)\,, \qquad (630)
\end{aligned}
$$

where the series in the exponential has been truncated at the second term due to the condition $\alpha > 3/4$ (see the equivalent section of the scalar calculations, (470)). Thus $\mathcal{F}_-^<$ becomes for small k (with $q \to p$)

$$
\begin{aligned}
\mathcal{F}_-^< = -\frac{ie^2}{\hbar}\int dtdt'\int_{k<\hbar^{\alpha-1}\lambda}\frac{d^3k}{2k(2\pi)^3}\frac{m^2}{p_0^2}\theta(t-t')\frac{p_0}{E_p(t)}\frac{p_0}{E_p(t')}\\
\times\left[\bar{u}_\alpha(p,t)\gamma^\mu u^\gamma(p,t)\bar{u}_\gamma(p,t')\gamma_\mu u_\alpha(p,t')\right]\exp\left(ik(t'-t)-\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})\right)\,.
\end{aligned}
\qquad (631)
$$

Let us denote the spinor combination in the square brackets above by

$$
\mathcal{S}^<(t,t') = \bar{u}_\alpha(p,t)\gamma^\mu u^\gamma(p,t)\bar{u}_\gamma(p,t')\gamma_\mu u_\alpha(p,t')\,. \qquad (632)
$$

In line with (630), we consider only the lowest order terms, i.e. $\hbar^0$ terms, for this spinor combination. At the end of the calculation, we shall note that higher order terms, of order $\hbar$ and above, do not contribute here. Firstly, we have the factor $\bar{u}_\alpha(p,t)\gamma^\mu u_\gamma(p,t)$. From the derivations for the equal time spinor identities in Appendix A[11], we write

$$
\bar{u}_\alpha(p)\gamma^0 u_\gamma(p) = \frac{E_p}{m}\delta_{\alpha\gamma} + \mathcal{O}(\hbar)\,, \qquad (633)
$$

and

$$
\bar{u}_\alpha(p)\boldsymbol{\gamma} u_\alpha(p) = \frac{\tilde{\mathbf{p}}}{m}\delta_{\alpha\gamma} + \mathcal{O}(\hbar)\,. \qquad (634)
$$

The components of the other factor, $\bar{u}_\gamma(p,t')\gamma_\mu u_\alpha(p,t')$, can then be obtained by reversing the spin indices and using the notation $\tilde{\mathbf{p}}' = \mathbf{p} - \mathbf{V}(t')$, $E_p' =$

[11]The particle equal time spinor identities are in section 4.1 of Appendix A.

$\sqrt{\tilde{\mathbf{p}}'^2 + m^2}$. Combining the identities and summing over the index γ, we obtain

$$\mathcal{S}^< = \frac{E_p E_p'}{m^2} - \frac{\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}'}{m^2} + \mathcal{O}(\hbar) \, . \tag{635}$$

We note straight away that the order $\hbar^0$ term here gives the same contribution as the scalar case: With $\tilde{p}(t) = (E_q(t), \tilde{\mathbf{q}}(t))$, we have

$$\begin{aligned}
&\Re \mathcal{F}_-^<(\mathbf{p}) \\
&= -\frac{ie^2}{\hbar} \int dt dt' \int_{k<\hbar^{\alpha-1}\lambda} \frac{d^3k}{2k(2\pi)^3} \frac{m^2}{E_p(t)E_p(t')} \theta(t-t') \left[\frac{\tilde{p}^\mu(t)}{m} \cdot \frac{\tilde{p}_\mu(t')}{m} \right] e^{ik\cdot(x'-x)} \\
&= -\frac{ie^2}{\hbar} \int dt dt' \int_{k<\hbar^{\alpha-1}\lambda} \frac{d^3k}{2k(2\pi)^3} \theta(t-t') \frac{dx^\mu}{dt} \frac{dx_\mu}{dt'} e^{ik\cdot(x'-x)} \\
&= -\frac{ie^2}{\hbar} \int d\xi d\xi' \int_{k<\hbar^{\alpha-1}\lambda} \frac{d^3k}{2k(2\pi)^3} \theta(\xi-\xi') \frac{dx^\mu}{d\xi} \frac{dx_\mu}{d\xi'} e^{ik(\xi'-\xi)} \, .
\end{aligned} \tag{636}$$

and thus

$$\Re \mathcal{F}_-^<(\mathbf{p}) = -\frac{e^2\lambda}{16\pi^3\hbar^{2-\alpha}} \int d\Omega dt \frac{1-\mathbf{v}^2}{1-\mathbf{n}\cdot\mathbf{v}} \, , \tag{637}$$

We can thus see that any terms of order $\hbar$ from the spinor combination would produce a contribution to the position shift of order $\hbar^\alpha$ and thus would not contribute here, in the $\hbar \to 0$ limit, as stated before.

We now show that the above expression can be arrived at from the leading order term of $\mathcal{F}_-$ in (624). Consequently, the higher order f_{nd} terms (which we recall do not contribute above the cut-off) do not contribute to the real part of $\mathcal{F}_-$ to order $\hbar^{-1}$. The leading order term, below the cut-off, we denote $\mathcal{F}_-^{<,0}$. From (624) we can write

$$\mathcal{F}_-^{<,0,1}(\mathbf{p}) = -\frac{e^2}{\hbar^2} \int_{K \le \hbar^\alpha \lambda} \frac{d^3\mathbf{K}}{(2\pi)^3 2K} dt \frac{m^2}{E_p E_q} \frac{f_{00}}{(-E_p + E_q + K)} \, . \tag{638}$$

In the $K \to 0$ limit, with $q \to p$, we have

$$f_{00} \to 2 - \frac{E_p^2}{m^2} + \frac{\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}}{m^2} \, . \tag{639}$$

thus

$$\mathcal{F}_-^{<,0}(\mathbf{p}) = -\frac{e^2}{\hbar^2}\int_{K\leq\hbar^\alpha\lambda}\frac{d^3\mathbf{K}}{(2\pi)^3 2K}dt\frac{m^2}{E_p^2} \times \frac{1}{(-E_p+E_q+K)}\left[2-\frac{\tilde{p}\cdot\tilde{p}}{m^2}\right]. \quad (640)$$

In the small K limit we have

$$-E_p+E_q+K \approx K - \mathbf{v}(t)\cdot\mathbf{K}, \quad (641)$$

where $\mathbf{v}(t) = \tilde{\mathbf{p}}/E_p = [\mathbf{p}-\mathbf{V}(t)]/E_p$ is the velocity of a particle with final momentum p. Thus splitting the $\mathbf{K}$ integral into spherical polars and integrating K gives

$$\begin{aligned}\mathcal{F}_-^{<,0}(\mathbf{p}) &= -\frac{e^2}{2(2\pi)^3\hbar^2}\int_{K\leq\hbar^\alpha\lambda}\frac{dK d\Omega K^2}{K}dt\frac{1-\mathbf{v}^2}{K(1-\mathbf{n}\cdot\mathbf{v})}\\ &= -\frac{e^2\lambda}{16\pi^3\hbar^{2-\alpha}}\int dt d\Omega\frac{1-\mathbf{v}^2}{(1-\mathbf{n}\cdot\mathbf{v})}, \end{aligned} \quad (642)$$

as required.

4.2. Antiparticle Loop. The antiparticle loop contribution to the forward scattering is represented by $\mathcal{F}_+$ in (599):

$$\mathcal{F}_+(\mathbf{p}) = \frac{ie^2m}{\hbar p_0}\int\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{d^3\mathbf{q}m}{(2\pi\hbar)^3 q_0}d^4x d^4x'\theta(t-t') \times \bar{\Phi}_\alpha(\mathbf{p}',x')\gamma^\nu\Psi_\beta(\mathbf{q},x')\bar{\Psi}^\beta(\mathbf{q},x)\gamma^\mu\Phi_\alpha(\mathbf{p},x)e^{-ik\cdot(x-x')}. \quad (599)$$

The spinor and electromagnetic mode functions can be expanded to give

$$\begin{aligned}&\bar{\Phi}_\alpha(\mathbf{p}',x')\gamma^\mu\Psi_\beta(\mathbf{q},x')\bar{\Psi}^\beta(\mathbf{q},x)\gamma_\mu\Phi_\alpha(\mathbf{p},x)e^{-ik\cdot(x-x')}\\ &= \left[\bar{u}_\alpha(p',t')\gamma^\mu v_\beta(q,t')\bar{v}^\beta(q,t)\gamma_\mu u_\alpha(p,t)\right]\\ &\quad\times\phi_{p'}^*(t')\bar{\phi}_q^*(t')\bar{\phi}_q(t)\phi_p(t)e^{-i\mathbf{p}'\cdot\mathbf{x}'}e^{-i\mathbf{q}\cdot\mathbf{x}'}e^{i\mathbf{q}\cdot\mathbf{x}}e^{i\mathbf{p}\cdot\mathbf{x}}e^{-ik\cdot(x-x')}. \end{aligned} \quad (643)$$

The spatial integrals produce the delta functions $(2\pi\hbar)^6\delta^3(\mathbf{p}+\mathbf{q}+\mathbf{K})\delta^3(\mathbf{p}-\mathbf{p}')$. Integrating over $\mathbf{p}'$ we write the antiparticle loop contribution in analogy to

the particle loop as

$$\mathcal{F}_{+}(\mathbf{p}) = \frac{ie^2 m}{\hbar p_0} \int \frac{d^3\mathbf{p}'}{(2\pi\hbar)^3} \frac{d^3\mathbf{k}}{2k(2\pi)^3} \frac{d^3\mathbf{q}m}{(2\pi\hbar)^3 q_0} d^4x d^4x' \theta(t-t') \times [\mathcal{S}_{+}]\, \phi^*_{p'}(t')\bar{\phi}^*_q(t')\bar{\phi}_q(t)\phi_p(t) e^{-ik(t-t')}\,, \tag{644}$$

where the antiparticle loop semiclassical spinor combination is defined as[12]

$$\mathcal{S}_{+} = \bar{u}_\alpha(p',t')\gamma^\mu v_\beta(q,t')\bar{v}^\beta(q,t)\gamma_\mu u_\alpha(p,t)\,. \tag{645}$$

As we did for the particle loop, we use the spinor identities from the appendix to obtain an expression for $\mathcal{S}_{+}$. We shall also use the same time variable transformation

$$t = \bar{t} - \frac{\hbar}{2}\eta \tag{646}$$

$$t' = \bar{t} + \frac{\hbar}{2}\eta\,, \tag{647}$$

and expand in terms of $\hbar$. It is important to note the two differences between $\mathcal{S}_{+}$ and $\mathcal{S}_{-}$. Firstly, and most obviously, is the presence of the antiparticle (negative energy) spinors as the inner spinors forming the loop part. Further to this we note that the order of the time variables is reversed. The time order reversal is, in terms of the $(\bar{t},\eta)$ variables, simply the transformation $\eta \to -\eta$. Consequently, from the inner spinors we obtain, to order $\hbar$

$$v_\alpha(q,t')\bar{v}^\alpha(q,t) = \frac{\gamma\cdot\tilde{q}_{+} - m}{2m} - \left(\frac{\hbar}{4mE^2_{q_+}} - \frac{i\hbar\eta}{4mE_{q_+}}\right)\gamma^5\boldsymbol{\gamma}\cdot\tilde{\mathbf{q}}_{+}\times\dot{\tilde{\mathbf{q}}}_{+} - \left(\frac{i\hbar}{4E^2_{q_+}} + \frac{\hbar\eta}{4E_{q_+}}\right)\gamma^0\boldsymbol{\gamma}\cdot\dot{\tilde{\mathbf{q}}}_{+}\,, \tag{648}$$

where we recall the definitions $\tilde{\mathbf{q}}_{+} = \mathbf{q} + \mathbf{V}(t)$ and $E_{q_+} = \sqrt{\tilde{\mathbf{q}}^2_{+} + m^2}$. Thus with the contracted gamma matrices,

$$\gamma^\mu v_\alpha(q,t')\bar{v}^\alpha(q,t)\gamma_\mu = -2 - \frac{\gamma\cdot\tilde{q}_{+}}{m} - \left(\frac{\hbar}{2mE^2_{q_+}} - \frac{i\hbar\eta}{2mE_{q_+}}\right)\gamma^5\boldsymbol{\gamma}\cdot\tilde{\mathbf{q}}_{+}\times\dot{\tilde{\mathbf{q}}}_{+}\,. \tag{649}$$

[12] We also use analogous terminology to that for $\mathcal{S}_{-}$ to refer to the inner and outer spinors. In this case, the inner spinors are the anti-particle spinors.

The antiparticle loop contribution is thus written

$$
\begin{aligned}
\mathcal{S}_+ &= \bar{u}_\alpha(p,t')\gamma^\mu v_\beta(q,t')\bar{v}^\beta(q,t)\gamma_\mu u_\alpha(p,t) \\
&= -2\left[\bar{u}_\alpha(p,t')u_\alpha(p,t)\right] - \frac{E_{q+}}{m}\left[\bar{u}_\alpha(p,t')\gamma^0 u_\alpha(p,t)\right] + \frac{\tilde{\mathbf{q}}_+}{m}\cdot\left[\bar{u}_\alpha(p,t')\boldsymbol{\gamma}u_\alpha(p,t)\right] \\
&\quad - \left(\frac{\hbar}{2mE_{q+}^2} - \frac{i\hbar\eta}{2mE_{q+}}\right)\left[\bar{u}_\alpha(p,t')\gamma^5\boldsymbol{\gamma}\cdot\tilde{\mathbf{q}}_+\times\dot{\tilde{\mathbf{q}}}_+ u_\alpha(p,t)\right]. \qquad (650)
\end{aligned}
$$

Substituting in the outer spinor identities (being careful with the η signs), we have

$$
\begin{aligned}
\mathcal{S}_+ &= -2\left[1 - \left(\frac{\hbar}{2mE_p} - \frac{i\hbar\eta}{2m}\right)\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p}\right] - \frac{E_{q+}}{m}\left[\frac{E_p}{m}\right] \\
&\quad + \frac{\tilde{\mathbf{q}}_+}{m}\cdot\left[\frac{\tilde{\mathbf{p}}}{m} - \left(\frac{\hbar}{2mE_p} - \frac{i\hbar\eta}{2m}\right)\left(\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\,\tilde{\mathbf{p}}}{E_p(E_p+m)} + \boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}} - \frac{\dot{E}_p\tilde{\mathbf{p}}}{E_p+m}\right)\right)\right] \\
&\quad + \left(\frac{\hbar}{2mE_{q+}^2} - \frac{i\hbar\eta}{2mE_{q+}}\right)\left[\boldsymbol{\xi}\cdot\tilde{\mathbf{q}}_+\times\dot{\tilde{\mathbf{q}}}_+ + \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{q}}_+\times\dot{\tilde{\mathbf{q}}}_+\,\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}}{m(E_p+m)}\right] \\
&= -2 - \frac{\tilde{q}_+\cdot\tilde{p}}{m^2} + \left(\frac{\hbar}{mE_p} + \frac{i\hbar\eta}{m}\right)\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p} \\
&\quad - \left(\frac{\hbar}{2m^2E_p} - \frac{i\hbar\eta}{2m^2}\right)\left(\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\,\tilde{\mathbf{p}}\cdot\tilde{\mathbf{q}}_+}{E_p(E_p+m)} + \tilde{\mathbf{q}}_+\cdot\boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}} - \frac{\dot{E}_p\tilde{\mathbf{p}}}{E_p+m}\right)\right) \\
&\quad + \left(\frac{\hbar}{2mE_{q+}^2} - \frac{i\hbar\eta}{2mE_{q+}}\right)\left[\boldsymbol{\xi}\cdot\tilde{\mathbf{q}}_+\times\dot{\tilde{\mathbf{q}}}_+ + \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{q}}_+\times\dot{\tilde{\mathbf{q}}}_+\,\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}}{m(E_p+m)}\right]. \qquad (651)
\end{aligned}
$$

The integrand of $\mathcal{F}_+$ is under both K and q integrals, but this time the delta function present is $\delta(\mathbf{p}+\mathbf{q}+\mathbf{K})$. We thus change the variables of integration via

$$\mathbf{q} \to -\mathbf{q} \qquad (652)$$

$$\mathbf{K} \to -\mathbf{K}\,. \qquad (653)$$

We then have the following relations:

$$
\begin{aligned}
&d^3\mathbf{q} \to d^3\mathbf{q} && d^3\mathbf{K} \to d^3\mathbf{K} \\
&\tilde{\mathbf{q}}_+ \to -\tilde{\mathbf{q}} && \dot{\tilde{\mathbf{q}}}_+ = -\dot{\tilde{\mathbf{q}}} \\
&E_{q+} \to E_q && K \to K\,,
\end{aligned}
\tag{654}
$$

along with the new delta function $\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{K})$, which is now the same as for $\mathcal{S}_-$. Labelling $\mathcal{S}_+$ under this transformation by $\mathcal{S}'_+$, we have

$$
\begin{aligned}
\mathcal{S}'_+ = -2 - \frac{E_q E_p}{m^2} - \frac{\tilde{\mathbf{q}}\cdot\tilde{\mathbf{p}}}{m^2} + \left(\frac{\hbar}{mE_p} + \frac{i\hbar\eta}{m}\right)\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p} \\
+ \left(\frac{\hbar}{2m^2E_p} - \frac{i\hbar\eta}{2m^2}\right)\left(\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\,\tilde{\mathbf{p}}\cdot\tilde{\mathbf{q}}}{E_p(E_p+m)} + \tilde{\mathbf{q}}\cdot\boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}} - \frac{\dot{E}_p\tilde{\mathbf{p}}}{E_p+m}\right)\right) \\
+ \left(\frac{\hbar}{2mE_q^2} - \frac{i\hbar\eta}{2mE_q}\right)\left[\boldsymbol{\xi}\cdot\tilde{\mathbf{q}}\times\dot{\tilde{\mathbf{q}}} + \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{q}}\times\dot{\tilde{\mathbf{q}}}\,\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}}{m(E_p+m)}\right].
\end{aligned}
\tag{655}
$$

For scalar semiclassical terms, we note $\bar{\phi}_{-q}(t) = \phi_q(t)$. Hence

$$
\phi^*_{p'}(t')\bar{\phi}^*_q(t')\bar{\phi}_q(t)\phi_p(t) \to \phi^*_{p'}(t')\phi^*_q(t')\phi_q(t)\phi_p(t)\,. \tag{656}
$$

For this scalar combination, we recall that

$$
\phi_{\mathbf{p}}(t) = \sqrt{\frac{p_0}{E_p(t)}}\exp\left[-\frac{i}{\hbar}\int_0^t E_p(t')\,dt'\right], \tag{657}
$$

and so we can write

$$
\begin{aligned}
&\phi^*_{p'}(t')\phi^*_q(t')\phi_q(t)\phi_p(t)e^{iK(t-t')/\hbar} \\
&\quad = |\phi_p(\bar{t})|^2|\phi_q(\bar{t})|^2\exp\left[i\int_{-\eta/2}^{\eta/2}\left(E_p(\bar{t}+\hbar\zeta) + E_q(\bar{t}+\hbar\zeta) + K\right)\,d\zeta\right].
\end{aligned}
\tag{658}
$$

Again, we can change the variable to help with the integration over η and define

$$
\left(E_p(t) + E_q(t) + K\right)\beta = \int_{-\eta/2}^{\eta/2}\left(E_p(t+\hbar\zeta) + E_q(t+\hbar\zeta) + K\right)\,d\zeta\,. \tag{659}
$$

As we recall from the scalar antiparticle loop, which used the same variable change at this point, there is no infrared divergence problem resulting from the β integration, so the previous manipulations, including consideration of

the higher order terms, are not needed here. The equations for η and $d\eta$ were given in the scalar work as

$$\eta = \left[1 - \frac{1}{24}\frac{\ddot{E}_p(t) + \ddot{E}_q(t)}{E_p(t) + E_q(t) + K}\hbar^2\beta^2 + \mathcal{O}(\hbar^4\beta^4)\right]\beta\,, \tag{451}$$

and

$$d\eta = \left[1 - \frac{1}{8}\frac{\ddot{E}_p(t) + \ddot{E}_q(t)}{E_p(t) + E_q(t) + K}\hbar^2\beta^2 + \mathcal{O}(\hbar^4\beta^4)\right]d\beta\,. \tag{452}$$

In this case we can ignore the higher order $\hbar^2$ terms.

Using the above, and changing notation $\bar{t} \to t$, we write $\mathcal{F}_+$ as

$$\mathcal{F}_+(\mathbf{p}) = \frac{ie^2 m}{p_0}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{d^3\mathbf{q}m}{(2\pi\hbar)^3 q_0}dtd\beta\theta(-\beta)(2\pi\hbar)^3\delta^3(\mathbf{p}-\mathbf{q}-\mathbf{K})$$
$$\times\left[\mathcal{S}'_+\right]|\phi_p(\bar{t})|^2|\phi_q(\bar{t})|^2\exp\left[i\left(E_p + E_q + K\right)\beta\right]\,. \tag{660}$$

As with the particle spinor combination, we produce an 'effective' version of $\mathcal{S}'_+$ under the delta function. The previous argument relating to the preferred direction of the axis for the $\mathbf{K}$ integration is still valid and thus we use the transformation

$$\tilde{\mathbf{q}} \to \tilde{\mathbf{p}}\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right)\,, \tag{661}$$

along with $\dot{\tilde{\mathbf{q}}} = \dot{\tilde{\mathbf{p}}}$. Proceeding as such, and recalling that we now have $\eta \to \beta$ we have

$$\begin{aligned}
\mathcal{S}'^{\text{eff}}_{+} &= -2 - \frac{E_q E_p}{m^2} - \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{p}}}{m^2}\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) + \left(\frac{\hbar}{mE_p} - \frac{i\hbar\beta}{m}\right)\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p} \\
&\quad - \left(\frac{\hbar}{2m^2E_p} - \frac{i\hbar\beta}{2m^2}\right)\left(-\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\,\tilde{\mathbf{p}}^2}{E_p(E_p+m)} - \tilde{\mathbf{p}}\cdot\boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}} - \frac{\dot{E}_p\tilde{\mathbf{p}}}{E_p+m}\right)\right) \\
&\quad \times\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) \\
&\quad + \left(\frac{\hbar}{2mE_q^2} - \frac{i\hbar\beta}{2mE_q}\right)\left[\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}} + \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{q}}}_{+}\,\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}}{m(E_p+m)}\right]\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) \\
&= -2 - \frac{E_q E_p}{m^2} - \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{p}}}{m^2}\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) + \left(\frac{\hbar}{mE_p} - \frac{i\hbar\beta}{m}\right)\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p} \\
&\quad - \left(\frac{\hbar}{2m^2E_p} - \frac{i\hbar\beta}{2m^2}\right)\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\frac{m}{E_p}\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) \\
&\quad + \left(\frac{\hbar}{2mE_q^2} - \frac{i\hbar\beta}{2mE_q}\right)\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left(1 - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right). \qquad (662)
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{S}'^{\text{eff}}_{+}(t,\beta) &= -2 - \frac{E_q E_p}{m^2} - \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{p}}}{m^2} + \frac{\hbar}{2m}\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left(\frac{1}{E_p^2} + \frac{1}{E_q^2}\right) \\
&\quad - \frac{i\hbar\beta}{2m}\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left(\frac{1}{E_p} + \frac{1}{E_q}\right) \\
&\quad + \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\left[\frac{\tilde{\mathbf{p}}^2}{m^2} + \boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left(\frac{\hbar}{2m}\left(\frac{1}{E_p^2} - \frac{1}{E_q^2}\right) - \frac{i\hbar\beta}{2m}\left(\frac{1}{E_p} - \frac{1}{E_q}\right)\right)\right]. \qquad (663)
\end{aligned}$$

Returning to the expression for $\mathcal{F}_+$, we recall that $|\phi_q(t)|^2 = q_0/E_q + \mathcal{O}(\hbar^2)$ and integrate out the delta function to produce

$$\begin{aligned}
\mathcal{F}_{+}(\mathbf{p}) = \frac{ie^2m^2}{\hbar^2 p_0}\int dt\frac{d^3\mathbf{K}}{(2\pi)^3 2KE_q}d\beta|\phi_p(\bar{t})|^2\theta(-\beta) \\
\times\left[\mathcal{S}'^{\text{eff}}_{+}(t,\beta)\right]\exp\left[i\left(E_p + E_q + K\right)\beta\right], \qquad (664)
\end{aligned}$$

where we have changed the variable from $\mathbf{k}$ to $\mathbf{K}$ in the measure[13] and we can now consider the definition of E_q to be $E_q = \sqrt{(\tilde{\mathbf{p}} - \mathbf{K})^2 + m^2}$. Integrating

[13]thus acquiring an $1/\hbar^2$ prefactor,

over β using the convergence factor, as in the particle loop case in (623), we find

$$\mathcal{F}_{+}(\mathbf{p}) = \frac{e^2m^2}{\hbar^2 p_0}\int dt \frac{d^3\mathbf{K}}{(2\pi)^3 2KE_q}|\phi_p(\bar{t})|^2\left[\frac{\left(f_{00}^{+}+\hbar f_{01}^{+}\right)}{(E_p+E_q+K)}+\frac{i\hbar f_{11}^{+}}{(E_p+E_q+K)^2}\right], \tag{665}$$

with

$$f_{00}^{+} = -2 - \frac{E_qE_p}{m^2} - \frac{\tilde{\mathbf{p}}\cdot\tilde{\mathbf{p}}}{m^2}\left(1-\frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\right) \tag{666}$$

$$f_{10}^{+} = \frac{1}{2m}\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left[\left(\frac{1}{E_p^2}+\frac{1}{E_q^2}\right)+\frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\left(\frac{1}{E_p^2}-\frac{1}{E_q^2}\right)\right] \tag{667}$$

$$f_{11}^{+} = \frac{i}{2m}\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\left[-\left(\frac{1}{E_p}+\frac{1}{E_q}\right)-\frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\left(\frac{1}{E_p}-\frac{1}{E_q}\right)\right]. \tag{668}$$

As a recap, the contributions to the real part of the forward scattering are the leading and first order $\hbar$ terms from both the particle and antiparticle loops as given in (624) and (665). The higher order contributions from (624) do not contribute to the real part to order $\hbar^{-1}$. As with the scalar case, we now renormalise the mass and calculate the contribution from the mass counterterm towards the renormalised forward scattering.

4.3. Mass renormalisation. The contribution to the forward scattering due to the renormalisation with the mass counterterm δm is given by

$$\delta\mathcal{F} = \frac{m}{\hbar p_0}\int dt|\phi_p(t)|^2\bar{u}_\alpha(p)\delta m u_\alpha(p)\,, \tag{669}$$

where the mass counter term itself is given by

$$\delta m = \Sigma(p)|_{\not{p}=m_P}\,, \tag{670}$$

and we define

$$\delta m_t = \bar{u}_\alpha(p)\Sigma(p)|_{\not{p}=m_P}u_\alpha(p)\,. \tag{671}$$

The self energy $\Sigma(p)$ (not to be confused with the term $\Sigma(t)$ used for the spinors), is calculated for the fermions in the absence of the potential and represents the loop given by the fermion and photon. The outer spinors found in $\delta\mathcal{F}$ however are the semiclassical spinors used in the presence of the classical

non-perturbative potential. To obtain the mass counterterm the self-energy is evaluated on the mass shell, i.e. with $p = p(t)$ and $\not{p} = m_P$. We recall that the counterterm is thus independent of the momentum. Hence δm is calculated using the standard QED without the external potential, but δm_t depends on $\mathbf{V}(t)$ through the time-dependent momenta of the spinors u_α. Using the Feynman rules for the free field, we have

$$\begin{aligned}\Sigma(p) &= -\frac{ie^2}{\hbar}\int \frac{d^4K}{(2\pi)^4}\frac{g_{\mu\nu}}{K^2+i\epsilon}\gamma^\mu \frac{1}{\not{p}-\not{K}-m+i\epsilon}\gamma^\nu \\ &= -\frac{ie^2}{\hbar}\int \frac{d^4K}{(2\pi)^4}\gamma^\mu \frac{\not{p}-\not{K}-m}{(K^2+i\epsilon)\left((p-K)^2-m^2+i\epsilon\right)}\gamma_\mu \\ &= -\frac{ie^2}{\hbar}\int \frac{d^4K}{(2\pi)^4}\frac{-2(\not{p}-\not{K})+4m}{(K^2+i\epsilon)\left((p-K)^2-m^2+i\epsilon\right)}\,,\end{aligned} \tag{672}$$

which is taken to be evaluated in the limit $\epsilon \to 0^+$. We shall complete the k_0 portion of the integration by complex contour integration. Let us define $\omega = \sqrt{|\mathbf{p}-\mathbf{K}|^2+m^2}$ and rewrite the denominator as

$$(K_0-|\mathbf{K}|+i\delta)(K_0+|\mathbf{K}|-i\delta)(K_0-p_0+\omega-i\delta)(K_0-p_0-\omega-i\delta)\,, \tag{673}$$

with the limit $\delta \to 0^+$. The poles in the upper half plane are

$$K_0 = -|\mathbf{K}|+i\delta\,;\; K_0 = p_0-\omega+i\delta\,. \tag{674}$$

Enclosing these poles by the contour (anticlockwise) the residue theorem gives us

$$\begin{aligned}&\Sigma(p) \\ &= -\frac{ie^2}{\hbar}(2\pi i)\int \frac{d^3\mathbf{K}}{(2\pi)^4}\left[\left.\frac{-2(\not{p}-\not{K})+4m}{(K_0-|\mathbf{K}|+i\delta)\left((p_0-K_0)^2-\omega^2+i\epsilon\right)}\right|_{K_0=-|\mathbf{K}|+i\delta}\right. \\ &\quad\left. + \left.\frac{-2(\not{p}-\not{K})+4m}{(K_0^2-|\mathbf{K}|^2+i\epsilon)(K_0-p_0-\omega-i\delta)}\right|_{K_0=p_0-\omega+i\delta}\right]\,.\end{aligned} \tag{675}$$

Making the substitutions for K_0, defining $K = |\mathbf{K}|$, and taking the appropriate limits, we obtain

$$\begin{aligned}\Sigma(p) &= \frac{e^2}{\hbar}\int\frac{d^3\mathbf{K}}{(2\pi)^3}\\ &\left[\frac{4m - 2\gamma^0\left(p_0+K\right) + 2\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{K})}{(-2K)\left((p_0+K)^2-\omega^2\right)} + \frac{4m - 2\gamma^0\omega + 2\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{K})}{\left((p_0-\omega)^2-K^2\right)(-2\omega)}\right]\\ &= -\frac{e^2}{\hbar}\int\frac{d^3\mathbf{K}}{(2\pi)^3}\left[4m\left[\frac{1}{2K\left((p_0+K)^2-\omega^2\right)} + \frac{1}{2\omega\left((p_0-\omega)^2-K^2\right)}\right]\right.\\ &\quad -2\gamma^0\left[\frac{p_0+K}{2K\left((p_0+K)^2-\omega^2\right)} + \frac{\omega}{2\omega\left((p_0-\omega)^2-K^2\right)}\right]\\ &\quad \left.+2\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{K})\left[\frac{1}{2K\left((p_0+K)^2-\omega^2\right)} + \frac{1}{2\omega\left((p_0-\omega)^2-K^2\right)}\right]\right].\end{aligned} \tag{676}$$

At this point, we make a short aside to note the following

$$\frac{1}{(p_0+K)^2-\omega^2} = \frac{1}{2\omega}\left(\frac{1}{p_0-\omega+K} - \frac{1}{p_0+\omega+K}\right) \tag{677}$$

$$\frac{1}{(p_0-\omega)^2-K^2} = \frac{1}{2K}\left(\frac{1}{p_0-\omega-K} - \frac{1}{p_0-\omega+K}\right) \tag{678}$$

$$\frac{p_0+K}{\left((p_0+K)^2-\omega^2\right.} = \frac{1}{2}\left(\frac{1}{p_0-\omega+K} + \frac{1}{p_0+\omega+K}\right). \tag{679}$$

Using these expansions, we can now write

$$\begin{aligned}\Sigma(p) &= -\frac{e^2}{\hbar}\int\frac{d^3\mathbf{K}m}{(2\pi)^3 2K\omega}\\ &\left[2\left[\frac{1}{p_0-\omega-K} - \frac{1}{p_0+\omega+K}\right] - \frac{\gamma^0\omega}{m}\left[\frac{1}{p_0-\omega-K} + \frac{1}{p_0+\omega+K}\right]\right.\\ &\quad\left.+\frac{\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{K})}{m}\left[\frac{1}{p_0-\omega-K} - \frac{1}{p_0+\omega+K}\right]\right].\end{aligned} \tag{680}$$

Evaluating the self-energy on the mass shell using the particle (as opposed to antiparticle) momentum $\tilde{\mathbf{p}} = \mathbf{p} - \mathbf{V}(t)$, we have $p_0 \to E_p(t)$ and $\omega \to E_q(t)$. As with the scalar counterterm, the mass shell uses the physical mass m_P, however given that the counterterm is again of order e^2, for our calculations

we may use m and there is no difference in the result at this order. Thus, with some rearrangement,

$$\Sigma(\tilde{p}) = -\frac{e^2}{\hbar}\int \frac{d^3\mathbf{K}m}{(2\pi)^3 2KE_q}\left[\frac{1}{E_p - E_q - K}\left[2 - \frac{E_q\gamma^0}{m} + \frac{(\tilde{\mathbf{p}} - \mathbf{K})\cdot\boldsymbol{\gamma}}{m}\right]\right.$$
$$\left. + \frac{1}{E_p + E_q + K}\left[-2 - \frac{E_q\gamma^0}{m} - \frac{(\tilde{\mathbf{p}} - \mathbf{K})\cdot\boldsymbol{\gamma}}{m}\right]\right]. \tag{681}$$

Adding the semiclassical outer spinors, with the momentum $\tilde{p}$, the (time-dependent) mass counter term is

$$\begin{aligned}
&\delta m_t \\
&= \bar{u}_\alpha(\tilde{p})\Sigma(\tilde{p})u_\alpha(\tilde{p}) \\
&= -\frac{e^2}{\hbar}\int \frac{d^3\mathbf{K}m}{(2\pi)^3 2KE_q} \\
&\left[\frac{1}{E_p - E_q - K}\left[2[\bar{u}_\alpha(p)u_\alpha(p)] - \frac{E_q}{m}[\bar{u}_\alpha(p)\gamma^0 u_\alpha(p)] + \frac{(\tilde{\mathbf{p}} - \mathbf{K})}{m}\cdot[\bar{u}_\alpha(p)\boldsymbol{\gamma}u_\alpha(p)]\right]\right. \\
&\left. + \frac{1}{E_p + E_q + K}\left[-2[\bar{u}_\alpha(p)u_\alpha(p)] - \frac{E_q}{m}[\bar{u}_\alpha(p)\gamma^0 u_\alpha(p)] - \frac{(\tilde{\mathbf{p}} - \mathbf{K})}{m}\cdot[\bar{u}_\alpha(p)\boldsymbol{\gamma}u_\alpha(p)]\right]\right].
\end{aligned} \tag{682}$$

Using the first order identities we have

$$\begin{aligned}
\delta m_t = &-\frac{e^2}{\hbar}\int \frac{d^3\mathbf{K}m}{(2\pi)^3 2KE_q} \\
&\left\{\frac{1}{E_p - E_q - K}\left[2\left(1 - \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p^2}\right) - \frac{E_q}{m}\frac{E_p}{m}\right.\right. \\
&\left. + \frac{(\tilde{\mathbf{p}} - \mathbf{K})}{m}\cdot\left(\frac{\tilde{\mathbf{p}}}{m} - \frac{\hbar}{2mE_p}\left(\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\,\tilde{\mathbf{p}}}{E_p\left(E_p + m\right)} + \boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}} - \frac{\dot{E}_p\tilde{\mathbf{p}}}{E_p + m}\right)\right)\right)\right] \\
&+ \frac{1}{E_p + E_q + K}\left[-2\left(1 - \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p^2}\right) - \frac{E_q}{m}\frac{E_p}{m}\right. \\
&\left.\left. - \frac{(\tilde{\mathbf{p}} - \mathbf{K})}{m}\cdot\left(\frac{\tilde{\mathbf{p}}}{m} - \frac{\hbar}{2mE_p}\left(\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\,\tilde{\mathbf{p}}}{E_p\left(E_p + m\right)} + \boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}} - \frac{\dot{E}_p\tilde{\mathbf{p}}}{E_p + m}\right)\right)\right)\right]\right\}.
\end{aligned} \tag{683}$$

We again make use of the symmetry present and choose the axis for integration along the direction $\tilde{\mathbf{p}}$ thus transforming $\mathbf{K} \to \left(\mathbf{K}\cdot\tilde{\mathbf{p}}/\tilde{\mathbf{p}}^2\right)\tilde{\mathbf{p}}$. Simplifying, we

obtain

$$
\begin{aligned}
\delta m_t = -\frac{e^2}{\hbar}\int \frac{d^3\mathbf{K}m}{(2\pi)^3 2KE_q} \\
\Bigg\{\Bigg[2 - \frac{E_q E_p}{m^2} + \frac{\tilde{\mathbf{p}}^2}{m^2} - \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p^2} - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\left[\frac{\tilde{\mathbf{p}}^2}{m^2} + \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p^2}\right]\Bigg] \\
\times \frac{1}{E_p - E_q - K} \\
+ \Bigg[-2 - \frac{E_q E_p}{m^2} - \frac{\tilde{\mathbf{p}}^2}{m^2} + \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p^2} + \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\left[\frac{\tilde{\mathbf{p}}^2}{m^2} + \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p^2}\right]\Bigg] \\
\times \frac{1}{E_p + E_q + K}\Bigg\} \; .
\end{aligned}
\tag{684}
$$

The mass counter term contribution to the forward scattering is therefore

$$
\begin{aligned}
\mathcal{F}^{\delta m} = -\frac{e^2 m}{p_0 \hbar^2}\int dt \frac{d^3\mathbf{K}m}{(2\pi)^3 2KE_q}|\phi_p(t)|^2 \\
\times \Bigg\{\Bigg[2 - \frac{E_q E_p}{m^2} + \frac{\tilde{\mathbf{p}}^2}{m^2} - \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p^2} - \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\left[\frac{\tilde{\mathbf{p}}^2}{m^2} + \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p^2}\right]\Bigg] \\
\frac{1}{E_p - E_q - K} \\
+ \Bigg[-2 - \frac{E_q E_p}{m^2} - \frac{\tilde{\mathbf{p}}^2}{m^2} + \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p^2} + \frac{\mathbf{K}\cdot\tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2}\left[\frac{\tilde{\mathbf{p}}^2}{m^2} + \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p^2}\right]\Bigg] \\
\frac{1}{E_p + E_q + K}\Bigg\} \; .
\end{aligned}
\tag{685}
$$

This counter term contribution is to be added to the leading and first order loop contributions to the forward scattering from (624) and (665). These two

contributions can be written

$$\mathcal{F}_{-} = \frac{e^2 m}{p_0 \hbar^2} \int dt \frac{d^3\mathbf{K} m}{(2\pi)^2 2K E_q} |\phi_p(t)|^2$$
$$\times \left\{ \left[2 - \frac{E_q E_p}{m^2} + \frac{\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}}{m^2} - \frac{\hbar}{2m} \boldsymbol{\xi} \cdot \tilde{\mathbf{p}} \times \dot{\tilde{\mathbf{p}}} \left(\frac{1}{E_p^2} + \frac{1}{E_q^2} \right) \right.\right.$$
$$\left. - \frac{\mathbf{K} \cdot \tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2} \left(\frac{\tilde{\mathbf{p}}^2}{m^2} + \frac{\hbar}{2m} \boldsymbol{\xi} \cdot \tilde{\mathbf{p}} \times \dot{\tilde{\mathbf{p}}} \left(\frac{1}{E_p^2} - \frac{1}{E_q^2} \right) \right) \right] \frac{1}{(E_p - E_q - K)}$$
$$\left. - \frac{\hbar}{2m} \boldsymbol{\xi} \cdot \tilde{\mathbf{p}} \times \dot{\tilde{\mathbf{p}}} \left[\left(\frac{1}{E_p} - \frac{1}{E_q} \right) + \frac{\mathbf{K} \cdot \tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2} \left(\frac{1}{E_p} + \frac{1}{E_q} \right) \right] \frac{1}{(E_p - E_q - K)^2} \right\}, \tag{686}$$

and

$$\mathcal{F}_{+} = \frac{e^2 m}{p_0 \hbar^2} \int dt \frac{d^3\mathbf{K} m}{(2\pi)^2 2K E_q} |\phi_p(t)|^2$$
$$\times \left\{ \left[-2 - \frac{E_q E_p}{m^2} - \frac{\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}}{m^2} + \frac{\hbar}{2m} \boldsymbol{\xi} \cdot \tilde{\mathbf{p}} \times \dot{\tilde{\mathbf{p}}} \left(\frac{1}{E_p^2} + \frac{1}{E_q^2} \right) \right.\right.$$
$$\left. + \frac{\mathbf{K} \cdot \tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2} \left(\frac{\tilde{\mathbf{p}}^2}{m^2} + \frac{\hbar}{2m} \boldsymbol{\xi} \cdot \tilde{\mathbf{p}} \times \dot{\tilde{\mathbf{p}}} \left(\frac{1}{E_p^2} - \frac{1}{E_q^2} \right) \right) \right] \frac{1}{(E_p + E_q + K)}$$
$$\left. + \frac{\hbar}{2m} \boldsymbol{\xi} \cdot \tilde{\mathbf{p}} \times \dot{\tilde{\mathbf{p}}} \left[\left(\frac{1}{E_p} + \frac{1}{E_q} \right) + \frac{\mathbf{K} \cdot \tilde{\mathbf{p}}}{\tilde{\mathbf{p}}^2} \left(\frac{1}{E_p} - \frac{1}{E_q} \right) \right] \frac{1}{(E_p + E_q + K)^2} \right\}, \tag{687}$$

where we have reordered the terms and for $\mathcal{F}_{-}$ brought the overall minus sign inside the integrand. Comparison between the mass counter term and the loops terms shows that the counter term cancels some, but not all of the loop contributions. From the derivation of the mass counter term it is notable that those terms coming from the time split are not present (as the self-energy is calculated at a particular time - note for the 'free' field, i.e. without the potential, the momentum at different times is the same in the absence of a further interaction). Additionally, the first order corrections to the inner spinors of the loop are also not present - the first order correction to the spinor is due to the presence of the time-dependent potential. Adding $\mathcal{F}_{-} + \mathcal{F}_{+} + \mathcal{F} + \mathcal{F}^{\delta m}$ we find the renormalised forward scattering contribution

is

$$\begin{aligned}\mathcal{F}^R(\mathbf{p}) = \frac{e^2 m}{p_0 \hbar^2} \int \frac{dt\, d^3\mathbf{K} m}{(2\pi)^3 2K E_q} |\phi_p(t)|^2 \frac{\hbar}{2m} \boldsymbol{\xi} \cdot \mathbf{p} \times \dot{\mathbf{p}} \\ \times \left\{ \frac{1}{E_q^2} \left(\frac{1}{E_p + E_q + K} - \frac{1}{E_p - E_q - K} \right) \left(1 - \frac{\mathbf{K} \cdot \mathbf{p}}{\mathbf{p}^2} \right) \right. \\ - \frac{1}{(E_p - E_q - K)^2} \left(\frac{1}{E_p} - \frac{1}{E_q} \right) + \frac{1}{(E_p + E_q + K)^2} \left(\frac{1}{E_p} + \frac{1}{E_q} \right) \\ \left. + \frac{\mathbf{K} \cdot \mathbf{p}}{\mathbf{p}^2} \left[-\frac{1}{(E_p - E_q - K)^2} \left(\frac{1}{E_p} + \frac{1}{E_q} \right) + \frac{1}{(E_p + E_q + K)^2} \left(\frac{1}{E_p} - \frac{1}{E_q} \right) \right] \right\} .\end{aligned} \tag{688}$$

where we have dropped the tilde notation on the $\mathbf{p}$ due to the fact that all the energy-momenta in the integrand are now the time-dependent elements of $(E_p, \mathbf{p} - \mathbf{V}(t))$. This contribution was not present in the scalar quantum position shift. We can regard this term as a 'correction' term leading to an additional contribution to the position shift when compared with either the scalar case or the classical case. As such, the term is in need of interpretation. Before doing so, however, further calculation and simplification of $\mathcal{F}^R$ will be useful.

4.3.1. *Integration and simplification.* Let us define $\mathcal{I}_K$ as the K integral part of the correction, viz

$$\mathcal{F}^R(\mathbf{p}) = \frac{e^2 m^2}{p_0 \hbar^2} \int dt |\phi_p(t)|^2 \frac{\hbar}{2m} \boldsymbol{\xi} \cdot \mathbf{p} \times \dot{\mathbf{p}}\, \mathcal{I}_K \,, \tag{689}$$

with

$$\begin{aligned}\mathcal{I}_K = \int \frac{d^3\mathbf{K}}{(2\pi)^3 2|\mathbf{K}| E_q} \left\{ \frac{1}{E_q^2} \left(\frac{1}{E_p + E_q + |\mathbf{K}|} - \frac{1}{E_p - E_q - |\mathbf{K}|} \right) \left(1 - \frac{\mathbf{K} \cdot \mathbf{p}}{\mathbf{p}^2} \right) \right. \\ - \frac{1}{(E_p - E_q - |\mathbf{K}|)^2} \left(\frac{1}{E_p} - \frac{1}{E_q} \right) + \frac{1}{(E_p + E_q + |\mathbf{K}|)^2} \left(\frac{1}{E_p} + \frac{1}{E_q} \right) \\ \left. + \frac{\mathbf{K} \cdot \mathbf{p}}{\mathbf{p}^2} \left[-\frac{1}{(E_p - E_q - |\mathbf{K}|)^2} \left(\frac{1}{E_p} + \frac{1}{E_q} \right) + \frac{1}{(E_p + E_q + |\mathbf{K}|)^2} \left(\frac{1}{E_p} - \frac{1}{E_q} \right) \right] \right\} .\end{aligned} \tag{690}$$

The reader may notice that we have returned to the original notation $|\mathbf{K}|$ for the modulus of the 3-vector. The reason for this is to prevent confusion in the

following calculation in which we shall need the 4-vector K. In order to help with the calculation $\mathcal{I}_K$, let us look at the result of the K_0 integration of the following two 4-dimensional integrals over K:

$$\mathcal{I}_A = \int \frac{d^4K}{(2\pi)^4 i} \frac{K_0}{(K^2 + i\varepsilon)\left((K-p)^2 - m^2 + i\varepsilon\right)^2}\,,$$
$$\mathcal{I}_B = \int \frac{d^4K}{(2\pi)^4 i} \frac{K_0}{(K^2 + i\varepsilon)^2\left((K-p)^2 - m^2 + i\varepsilon\right)}\,, \tag{691}$$

with the limit $\varepsilon \to 0^+$. To clarify the previous comment on the notation, we confirm that $K^2 = K^\mu K_\mu$ here and for the remainder of this section.

Firstly, for $\mathcal{I}_A$, we note that the denominator is

$$\begin{aligned}&(K_0^2 - |\mathbf{K}|^2 + i\varepsilon)\left((K_0 - E_p)^2 - |\mathbf{K} - \mathbf{p}|^2 - m^2 + i\varepsilon\right)^2 \\ &= (K_0^2 - |\mathbf{K}|^2 + i\varepsilon)\left((K_0 - E_p)^2 - E_q^2 + i\varepsilon\right)^2 \\ &= (K_0 + |\mathbf{K}| - i\delta)(K_0 - |\mathbf{K}| + i\delta)(K_0 - E_p - E_q + i\delta)^2(K_0 - E_p + E_q - i\delta)^2\,,\end{aligned} \tag{692}$$

with the limit $\delta \to 0^+$. The poles in the upper half plane are

$$K_0 = -|\mathbf{K}| + i\delta\,,$$
$$K_0 = E_p - E_q + i\delta\,, \tag{693}$$

where the second pole is second order. The residue for the singularity at $K_0 = -|\mathbf{K}| + i\delta$, in the $\delta \to 0^+$ limit, is

$$\frac{1}{2}\frac{1}{(E_p + E_q + |\mathbf{K}|)^2(E_p - E_q + |\mathbf{K}|)^2}\,, \tag{694}$$

and the residue at $K_0 = E_p - E_q + i\delta$ is

$$\frac{1}{4}\frac{E_p^3 - 4E_p^2E_q + 5E_pE_q^2 - 2E_q^3 - |\mathbf{K}|^2E_p}{E_q^3(E_p - E_q + |\mathbf{K}|)^2(E_p - E_q - |\mathbf{K}|)^2}\,. \tag{695}$$

The sum of the residues, after some rearranging, gives

$$\begin{aligned}-\frac{1}{8|\mathbf{K}|}\Bigg[&\frac{E_p + E_q}{(E_q + E_p + |\mathbf{K}|)^2E_q^2} + \frac{E_p - E_q}{(|\mathbf{K}| + E_q - E_p)^2E_q^2} \\ &+ \frac{E_p}{(|\mathbf{K}| + E_q - E_p)E_q^3} + \frac{E_p}{(|\mathbf{K}| + E_q + E_p)E_q^3}\Bigg]\,.\end{aligned} \tag{696}$$

Consequently, the K_0 integration of $\mathcal{I}_A$ gives

$$\mathcal{I}_A = -\int \frac{d^3\mathbf{K}}{(2\pi)^3 8|\mathbf{K}|}\left[\frac{E_p+E_q}{(E_q+E_p+|\mathbf{K}|)^2E_q^2}+\frac{E_p-E_q}{(|\mathbf{K}|+E_q-E_p)^2E_q^2}\right.$$
$$\left.+\frac{E_p}{(|\mathbf{K}|+E_q-E_p)E_q^3}+\frac{E_p}{(|\mathbf{K}|+E_q+E_p)E_q^3}\right]. \quad (697)$$

Secondly, we analogously consider $\mathcal{I}_B$. The denominator gives

$$(K_0+|\mathbf{K}|-i\delta)^2(K_0-|\mathbf{K}|+i\delta)^2(K_0-E_p-E_q+i\delta)(K_0-E_p+E_q-i\delta)\,. \quad (698)$$

The poles in the upper half plane are again

$$K_0 = -|\mathbf{K}|+i\delta\,,$$
$$K_0 = E_p-E_q+i\delta\,, \quad (699)$$

where this time the first pole is second order. For this integrand, the residue at $K_0=-|\mathbf{K}|+i\delta$, in the $\delta\to 0^+$ limit, is

$$-\frac{1}{2}\frac{E_p+|\mathbf{K}|}{|\mathbf{K}|(E_p+E_q+|\mathbf{K}|)^2(E_p-E_q+|\mathbf{K}|)^2}\,, \quad (700)$$

and the residue at $K_0=E_p-E_q+i\delta$ is

$$-\frac{1}{2}\frac{E_p-E_q}{E_q(E_p-E_q+|\mathbf{K}|)^2(E_p-E_q-|\mathbf{K}|)^2}\,. \quad (701)$$

The sum of the residues rearranges to produce

$$\frac{1}{8|\mathbf{K}|E_q}\left[\frac{1}{(E_p+E_q+|\mathbf{K}|)^2}-\frac{1}{(E_p-E_q-|\mathbf{K}|)^2}\right]. \quad (702)$$

The K_0 integration of $\mathcal{I}_B$ thus gives

$$\mathcal{I}_B = \int \frac{d^3\mathbf{K}}{(2\pi)^3 8|\mathbf{K}|E_q}\left[\frac{1}{(E_p+E_q+|\mathbf{K}|)^2}-\frac{1}{(E_p-E_q-|\mathbf{K}|)^2}\right]. \quad (703)$$

We now return to the correction integral. Expanding and rearranging $\mathcal{I}_K$ leads to the following

$$\begin{aligned}
\mathcal{I}_K = \int \frac{d^3\mathbf{K}}{(2\pi)^3 2|\mathbf{K}|} \Bigg\{ & \frac{1}{E_p E_q^3} \left(\frac{E_p}{E_p + E_q + |\mathbf{K}|} + \frac{E_p}{|\mathbf{K}| + E_q - E_p} \right) \\
& + \frac{E_p - E_q}{\left(E_p - E_q - |\mathbf{K}|\right)^2 E_q^2 E_p} + \frac{E_p + E_q}{\left(E_p + E_q + |\mathbf{K}|\right)^2 E_q^2 E_p} \\
& - \frac{\mathbf{K} \cdot \mathbf{p}}{\mathbf{p}^2} \Bigg[\frac{1}{E_p E_q^3} \left(\frac{E_p}{E_p + E_q + |\mathbf{K}|} + \frac{E_p}{|\mathbf{K}| + E_q - E_p} \right) \\
& + \frac{E_p + E_q}{\left(E_p - E_q - |\mathbf{K}|\right)^2 E_q^2 E_p} + \frac{E_p - E_q}{\left(E_p + E_q + |\mathbf{K}|\right)^2 E_q^2 E_p} \Bigg] \Bigg\} \\
= \frac{4}{E_p} \int \frac{d^3\mathbf{K}}{(2\pi)^3 8|\mathbf{K}|} \Bigg\{ & \frac{1}{E_q^3} \left(\frac{E_p}{E_p + E_q + |\mathbf{K}|} + \frac{E_p}{|\mathbf{K}| + E_q - E_p} \right) \\
& + \frac{E_p - E_q}{\left(E_p - E_q - |\mathbf{K}|\right)^2 E_q^2} + \frac{E_p + E_q}{\left(E_p + E_q + |\mathbf{K}|\right)^2 E_q^2} \\
& - \frac{\mathbf{K} \cdot \mathbf{p}}{\mathbf{p}^2} \Bigg[\frac{1}{E_q^3} \left(\frac{E_p}{E_p + E_q + |\mathbf{K}|} + \frac{E_p}{|\mathbf{K}| + E_q - E_p} \right) \\
& + \frac{E_p - E_q}{\left(E_p - E_q - |\mathbf{K}|\right)^2 E_q^2} + \frac{E_p + E_q}{\left(E_p + E_q + |\mathbf{K}|\right)^2 E_q^2} \\
& + \frac{2E_q}{\left(E_p - E_q - |\mathbf{K}|\right)^2 E_q^2} - \frac{2E_q}{\left(E_p + E_q + |\mathbf{K}|\right)^2 E_q^2} \Bigg] \Bigg\} \, . \qquad (704)
\end{aligned}$$

Using the above results for $\mathcal{I}_A$ and $\mathcal{I}_B$, we see that we can rewrite $\mathcal{I}_K$ in terms of the 4 dimensional integrals:

$$\begin{aligned}
\mathcal{I}_K = & -\frac{4}{E_p} \int \frac{d^4K}{(2\pi)^4 i} \frac{K_0}{(K^2 + i\varepsilon) \left((K-p)^2 - m^2 + i\varepsilon\right)^2} \\
& + \sum_i \frac{4p_i}{E_p \mathbf{p}^2} \int \frac{d^4K}{(2\pi)^4 i} \frac{K_0 K_i}{(K^2 + i\varepsilon) \left((K-p)^2 - m^2 + i\varepsilon\right)^2} \\
& + \sum_i \frac{8p_i}{E_p \mathbf{p}^2} \int \frac{d^4K}{(2\pi)^4 i} \frac{K_0 K_i}{(K^2 + i\varepsilon)^2 \left((K-p)^2 - m^2 + i\varepsilon\right)} \, . \qquad (705)
\end{aligned}$$

Let us now define the following 4-dimensional integrals (in Minkowski space):

$$\mathcal{I}^{(1)}_{\mu} = \int \frac{d^4K}{(2\pi)^4 i} \frac{K_\mu}{(K^2+i\varepsilon)((K-p)^2-m^2+i\varepsilon)^2}\,, \tag{706}$$

$$\mathcal{I}^{(2)}_{\mu\nu} = \int \frac{d^4K}{(2\pi)^4 i} \frac{K_\mu K_\nu}{(K^2+i\varepsilon)((K-p)^2-m^2+i\varepsilon)^2}\,, \tag{707}$$

$$\mathcal{I}^{(3)}_{\mu\nu} = \int \frac{d^4K}{(2\pi)^4 i} \frac{K_\mu K_\nu}{(K^2+i\varepsilon)^2((K-p)^2-m^2+i\varepsilon)}\,. \tag{708}$$

We then obtain

$$\mathcal{I}_K = \frac{4}{E_p}\left[-\mathcal{I}^{(1)}_0 + \sum_i \frac{p_i}{\mathbf{p}^2}\left(\mathcal{I}^{(2)}_{0i} + 2\mathcal{I}^{(3)}_{0i}\right)\right]\,. \tag{709}$$

We now proceed to calculate these 4-dimensional integrals. We start by noting the following identities, which we shall use in the calculation:

$$\frac{1}{ab^2} = \int_0^1 dy \frac{2y}{((1-y)a+yb)^3}\,, \tag{710}$$

$$\int_0^\infty \frac{x^{\mu-1}}{(x+\alpha)^\nu}dx = \alpha^{\mu-\nu}\frac{\Gamma(\nu-\mu)\Gamma(\mu)}{\Gamma(\nu)} \qquad \text{for positive } \alpha\,. \tag{711}$$

We shall also make use of the following time-coordinate rotation from integration in Minkowski K space to Euclidean K_E space:

$$K_4 = -iK_0\,, \tag{712}$$

$$K^2 = -(K_1^2+K_2^2+K_3^2+K_4^2) = -K_E^2\,, \tag{713}$$

$$d^4K_E = d^3K\,dK_4 = -id^4K\,. \tag{714}$$

For D-dimensional Euclidean coordinates, the integration of a function which is only dependent on the radial coordinate of the hyperspherical polar coordinates can be written

$$\int d^Dx f(r) = \int dr \frac{2\pi^{D/2}}{\Gamma(D/2)} r^{D-1} f(r) = \int d(r^2)\frac{\pi^{D/2}}{\Gamma(D/2)}(r^2)^{D/2-1}f(r)\,. \tag{715}$$

In 4 dimensions, this becomes

$$\int d^4x f(r) = \int dr \frac{2\pi^2}{\Gamma(2)} r^3 f(r) = \int d(r^2)\pi^2 r^2 f(r)\,. \tag{716}$$

4.3.2. *First integral.*

$$
\begin{aligned}
\mathcal{I}_\mu^{(1)} &= \int \frac{d^4K}{(2\pi)^4 i} \frac{K_\mu}{(K^2+i\varepsilon)((K-p)^2-m^2+i\varepsilon)^2} \\
&= \int \frac{d^4K}{(2\pi)^4 i} \int_0^1 dy \frac{2yK_\mu}{\left[(1-y)(K^2+i\varepsilon)+y((K-p)^2-m^2+i\varepsilon)\right]^3} \\
&= \int_0^1 dy \int \frac{d^4K}{(2\pi)^4 i} \\
&\quad \times \frac{2yK_\mu}{\left[K^2-yK^2+(1-y)i\varepsilon+yK^2+yp^2-2yK\cdot p-ym^2+yi\varepsilon\right]^3} \\
&= \int_0^1 dy \int \frac{d^4K}{(2\pi)^4 i} \frac{2yK_\mu}{\left[K^2-2yK\cdot p+y(p^2-m^2)+i\varepsilon\right]^3} \\
&= \int_0^1 dy \int \frac{d^4K}{(2\pi)^4 i} \frac{2yK_\mu}{\left[K^2-2yK\cdot p+i\varepsilon\right]^3} \, .
\end{aligned}
\tag{717}
$$

Let

$$
\kappa_\mu = K_\mu - yp_\mu \, . \tag{718}
$$

Then we have

$$
d^4K = d^4\kappa \, , \tag{719}
$$

and

$$
\begin{aligned}
\kappa^2 - y^2p^2 &= K^2 + y^2p^2 - 2yK\cdot p - y^2p^2 \\
&= K^2 - 2yK\cdot p \, .
\end{aligned}
\tag{720}
$$

Thus we can change variables to produce

$$
\mathcal{I}_\mu^{(1)} = \int_0^1 dy 2y \int \frac{d^4\kappa}{(2\pi)^4 i} \frac{\kappa_\mu + yp_\mu}{\left[\kappa^2 - y^2p^2 + i\varepsilon\right]^3} \, . \tag{721}
$$

Using $p^2 = m^2$, changing notation from $\kappa \to K$, we have

$$
\mathcal{I}_\mu^{(1)} = \int_0^1 dy 2y \int \frac{d^4K}{(2\pi)^4 i} \frac{K_\mu + yp_\mu}{\left[K^2 - y^2m^2 + i\varepsilon\right]^3} \, . \tag{722}
$$

The term proportional to K_μ in the numerator is odd and integrates to zero, we thus have

$$
\mathcal{I}_\mu^{(1)} = \int_0^1 dy 2y^2 p_\mu \int \frac{d^4K}{(2\pi)^4 i} \frac{1}{\left[K^2 - y^2m^2 + i\varepsilon\right]^3} \, . \tag{723}
$$

Rotating the time coordinate to produce 4-d Euclidean space, we find:

$$\mathcal{I}_{\mu}^{(1)} = -\int_0^1 dy 2y^2 p_\mu \int \frac{d^4K_E}{(2\pi)^4} \frac{1}{\left[K_E^2 + y^2m^2\right]^3} \,, \tag{724}$$

where we have taken the limit $\varepsilon \to 0^+$ as the integral converges. Note the overall minus sign and consequent rearrangement of the denominator of the integrand. Using 4-dimensional hyperspherical polar coordinates, we have:

$$\mathcal{I}_{\mu}^{(1)} = -\int_0^1 dy 2y^2 p_\mu \int_0^\infty \frac{d(K_E^2)}{(2\pi)^4} \frac{\pi^2 K_E^2}{\left[K_E^2 + y^2m^2\right]^3} \,. \tag{725}$$

Using (711), with $\mu = 2$ and $\nu = 3$, to perform the K_E^2 integral, we obtain

$$\begin{aligned}
\mathcal{I}_{\mu}^{(1)} &= -\int_0^1 dy 2y^2 p_\mu \frac{\pi^2}{(2\pi)^4 y^2 m^2} \frac{\Gamma(1)\Gamma(2)}{\Gamma(3)} \\
&= -\frac{\pi^2 p_\mu}{(2\pi)^4 m^2} \int_0^1 dy \frac{y^2}{y^2} \\
&= -\frac{\pi^2 p_\mu}{(2\pi)^4 m^2} \,.
\end{aligned} \tag{726}$$

4.3.3. *Second Integral.* The second integral follows the same method as the first:

$$\begin{aligned}
\mathcal{I}_{\mu\nu}^{(2)} &= \int \frac{d^4K}{(2\pi)^4 i} \frac{K_\mu K_\nu}{(K^2 + i\varepsilon)((K-p)^2 - m^2 + i\varepsilon)^2} \\
&= \int \frac{d^4K}{(2\pi)^4 i} \int_0^1 dy \frac{2yK_\mu K_\nu}{\left[(1-y)(K^2 + i\varepsilon) + y((K-p)^2 - m^2 + i\varepsilon)\right]^3} \\
&= \int_0^1 dy \int \frac{d^4K}{(2\pi)^4 i} \frac{2yK_\mu K_\nu}{\left[K^2 - 2yK\cdot p + y(p^2 - m^2) + i\varepsilon\right]^3} \\
&= \int_0^1 dy \int \frac{d^4K}{(2\pi)^4 i} \frac{2yK_\mu K_\nu}{\left[K^2 - 2yK\cdot p + i\varepsilon\right]^3} \,.
\end{aligned} \tag{727}$$

We again use the change of variables given by

$$\begin{aligned}
\kappa_\mu &= K_\mu - y p_\mu \,, \\
d^4K &= d^4\kappa \,,
\end{aligned} \tag{728}$$

and

$$\kappa^2 - y^2p^2 = K^2 - 2yK\cdot p \,, \tag{729}$$

to obtain

$$\mathcal{I}^{(2)}_{\mu\nu} = \int_0^1 dy 2y \int \frac{d^4\kappa}{(2\pi)^4 i} \frac{(\kappa_\mu + yp_\mu)(\kappa_\mu + yp_\mu)}{[\kappa^2 - y^2m^2 + i\varepsilon]^3} . \tag{730}$$

We change the notation as $\kappa \to K$ and expand the numerator. Those terms proportional to K_μ or K_ν odd functions of K_μ and K_ν, and integrate to zero. The term proportional to $K_\mu K_\nu$ is an odd function when $\mu \neq \nu$, which is the case that we require. Thus for the $\mu = 0$, $\nu = i$ elements, we have

$$\mathcal{I}^{(2)}_{0i} = \int_0^1 dy 2y^3 E_p p_i \int \frac{d^4K}{(2\pi)^4 i} \frac{1}{[K^2 - y^2m^2 + i\varepsilon]^3} . \tag{731}$$

The K integral can be recognised as the same as that in $\mathcal{I}^{(1)}_\mu$ in (723), thus we have

$$\begin{aligned}
\mathcal{I}^{(2)}_{0i} &= -\int_0^1 dy 2y^3 E_p p_i \frac{\pi^2}{(2\pi)^4 y^2 m^2} \frac{\Gamma(1)\Gamma(2)}{\Gamma(3)} \\
&= -\frac{\pi^2 E_p p_i}{(2\pi)^4 m^2} \int_0^1 dy \frac{y^3}{y^2} \\
&= -\frac{1}{2} \frac{\pi^2 E_p p_i}{(2\pi)^4 m^2} .
\end{aligned} \tag{732}$$

4.3.4. *Third Integral.* For the third integral we proceed using the same method as before, but note the change in the denominator (and thus the K integral) from the previous cases. We have

$$\begin{aligned}
\mathcal{I}^{(3)}_{\mu\nu} &= \int \frac{d^4K}{(2\pi)^4 i} \frac{K_\mu K_\nu}{(K^2 + i\varepsilon)^2((K-p)^2 - m^2 + i\varepsilon)} \\
&= \int \frac{d^4K}{(2\pi)^4 i} \int_0^1 dy \frac{2yK_\mu K_\nu}{[y(K^2 + i\varepsilon) + (1-y)((K-p)^2 - m^2 + i\varepsilon)]^3} \\
&= \int \frac{d^4K}{(2\pi)^4 i} \int_0^1 dy \frac{2yK_\mu K_\nu}{[K^2 - (1-y)2K \cdot p + i\varepsilon]^3} .
\end{aligned} \tag{733}$$

This time we change variables using

$$\kappa'_\mu = K_\mu - (1-y)p_\mu , \tag{734}$$

and

$$\kappa'^2 - (1-y)^2 p^2 = K^2 - (1-y)2K \cdot p . \tag{735}$$

Thus we have

$$\mathcal{I}^{(3)}_{\mu\nu} = \int_0^1 dy 2y \int \frac{d^4\kappa'}{(2\pi)^4 i} \frac{\left(\kappa'_\mu + (1-y)p_\mu\right)\left(\kappa'_\nu + (1-y)p_\nu\right)}{\left[\kappa'^2 - (1-y)^2 p^2 + i\varepsilon\right]^3} \,. \tag{736}$$

The current situation is analogous to the calculation for $\mathcal{I}^{(2)}_{\mu\nu}$. After changing the notation as $\kappa' \to K$, we can again remove the terms proportional to K_μ and K_ν. Similarly, we again require only the $\mu = 0$, $\nu = i$ terms, and thus can also remove the $K_0 K_i$ term. The remainder gives

$$\mathcal{I}^{(3)}_{0i} = \int_0^1 dy 2y(1-y)^2 E_p p_i \int \frac{d^4K}{(2\pi)^4 i} \frac{1}{[K^2 - (1-y)^2 m^2 + i\varepsilon]^3} \,. \tag{737}$$

Rotating the K_0 coordinate, we produce

$$\begin{aligned}
\mathcal{I}^{(3)}_{0i} &= -\int_0^1 dy 2y(1-y)^2 E_p p_i \int \frac{d^4K_E}{(2\pi)^4 i} \frac{1}{[K_E^2 + (1-y)^2 m^2]^3} \\
&= -\int_0^1 dy 2y(1-y)^2 E_p p_i \int \frac{d(K_E)^2}{(2\pi)^4} \frac{\pi^2 K_E^2}{[K_E^2 + (1-y)^2 m^2]^3} \\
&= -\int_0^1 dy 2y(1-y)^2 E_p p_i \frac{\pi^2}{(2\pi)^4 (1-y)^2 m^2} \frac{\Gamma(1)\Gamma(2)}{\Gamma(3)} \\
&= -\frac{\pi^2 E_p p_i}{(2\pi)^4 m^2} \int_0^1 dy y \\
&= -\frac{1}{2} \frac{\pi^2 E_p p_i}{(2\pi)^4 m^2} \,.
\end{aligned} \tag{738}$$

4.3.5. *Evaluation.* Collecting together the results, we have

$$\begin{aligned}
\mathcal{I}^{(1)}_0 &= -\frac{\pi^2 E_p}{(2\pi)^4 m^2} \,, \\
\mathcal{I}^{(2)}_{0i} &= -\frac{1}{2} \frac{\pi^2 E_p p_i}{(2\pi)^4 m^2} \,, \\
\mathcal{I}^{(3)}_{0i} &= -\frac{1}{2} \frac{\pi^2 E_p p_i}{(2\pi)^4 m^2} \,.
\end{aligned} \tag{739}$$

We recall that

$$\mathcal{I}_K = \frac{4}{E_p} \left[-\mathcal{I}^{(1)}_0 + \sum_i \frac{p_i}{\mathbf{p}^2} \left(\mathcal{I}^{(2)}_{0i} + 2\mathcal{I}^{(3)}_{0i} \right) \right] . \tag{740}$$

Thus

$$\mathcal{I}_K = -\frac{2\pi^2}{(2\pi)^4 m^2} \,. \tag{741}$$

The correction term for the forward scattering is then given by

$$\begin{aligned}\mathcal{F}^R(\mathbf{p}) &= \frac{e^2m^2}{p_0\hbar^2}\int dt|\phi_p(t)|^2\frac{\hbar}{2m}\,\boldsymbol{\xi}\cdot\mathbf{p}\times\dot{\mathbf{p}}\,\mathcal{I}_K \\ &= -\frac{e^2}{16\pi^2 p_0 m\hbar}\int dt|\phi_p(t)|^2\,\boldsymbol{\xi}\cdot\mathbf{p}\times\dot{\mathbf{p}}\,. \end{aligned} \tag{742}$$

With this simplified expression, we can now proceed to consider the interpretation of the correction.

4.4. Vertex correction. The external potential is regarded as classical, but is coupled to the spinor field via the interaction term $\gamma^\mu V_\mu$ and this vertex has an associated one-loop correction. The one-loop process thus not only alters the propagator, but also the interaction with the external field. This correction is well known and responsible for the anomalous magnetic moment.[14] Our external potential, although considered non-perturbatively, acts like a minimally substituted electromagnetic external field. As such, whilst the one-loop corrections to the vertices for the interaction between the electromagnetic field and the spinor field in the emission and one-loop forward scattering diagrams are of higher order in e^2 than those with which we are concerned and thus ignored, the external potential V is still coupled to the spinor field via the term $\gamma^\mu V_\mu$. We stress that this effect is simply one that the current one-loop forward scattering process has on the external field coupling, and not an additional process which we are now adding. We shall show that the one-loop correction to the vertex is entirely responsible for the 'correction' term $\mathcal{F}^R$ which we have found.

As we are simply interpreting $\mathcal{F}^R$, and not deriving the vertex correction from scratch, it will be sufficient to quote some of the relevant theory. The

[14]The anomalous magnetic moment at the one-loop level for QED was first derived by Schwinger [**36**]. It is currently the most accurately tested and confirmed prediction in the history of physics.

renormalised vertex is given by[15]

$$\Gamma^R_\rho(p', p) = \gamma_\rho F_1(q^2) + \frac{i}{2m}\sigma_{\rho\nu} q^\nu F_2(q^2)\,, \tag{743}$$

where F_i are the form factors, the evaluation of which for our circumstances is given shortly; p, p' are the momenta before and after the vertex and q is the momentum transfer $p' - p$. We also have $\sigma^{\rho\nu} = \frac{i}{2}[\gamma^\rho, \gamma^\nu]$. For the coupling of the spinor field to the external potential,

$$V^\rho(x) = \int \frac{d^4q}{(2\pi)^4} V^\mu(q) e^{-iq\cdot x}\,, \tag{744}$$

we have

$$\partial^\nu V^\rho(x) = \int \frac{d^4q}{(2\pi)^4}(-iq^\nu) V^\mu(q) e^{-iq\cdot x}\,. \tag{745}$$

The interaction of the classical potential can be regarded as taking place is the so-called quasi-static limit, $q \to 0$ and from the above, the momentum transfer is replaced by the derivative operator. Consequently, the coupling term $\overline{\psi}\gamma_\rho\psi\, V^\rho$ changes to

$$\overline{\psi}\gamma_\rho\psi\, F_1(-\hbar^2\partial^2)V^\rho - \frac{\hbar}{2m}\overline{\psi}\sigma_{\rho\nu}\psi F_2(-\hbar^2\partial^2)\partial^\nu V^\rho\,. \tag{746}$$

We are only interested in the lowest $\hbar$ order (renormalised) vertex corrections and it can be shown[16] that in our limit we thus obtain

$$\overline{\psi}\gamma_\rho\psi\, F_1(0)V^\rho - \frac{\hbar}{2m}\overline{\psi}\sigma_{\rho\nu}\psi F_2(0)\partial^\nu V^\rho = \overline{\psi}\gamma_\rho\psi\, V^\rho - \frac{\alpha_c}{4\pi m}\overline{\psi}\sigma_{\rho\nu}\psi\, \partial^\nu V^\rho\,. \tag{747}$$

We recall that $\alpha_c = e^2/4\pi$ and consequently, the second term of (747) gives our correction term as a result of including the renormalised vertex:

$$-\frac{e^2}{16\pi^2 m}\overline{\psi}\sigma_{\rho\nu}\psi\, \partial^\nu V^\rho\,. \tag{748}$$

It is this correction term in which we are interested. As in the case of the mass counter term, we could regard this correction as an interaction in the

[15] See for example $(7-54)$ on p340 of Itzykson and Zuber's Quantum Field Theory [**24**]. The theory discussed briefly here is given in more detail in [**24**] in p340-341 and p347 in particular.

[16] This calculation is performed in the referenced pages in [**24**] in the previous footnote.

Lagrangian producing a Feynman diagram contribution. We can then calculate the contribution of this term towards the forward scattering.[17]

Now, our external potential has only space components and they are only dependent on time t. The term (748) can thus be rewritten as

$$\frac{e^2}{16\pi^2 m}\overline{\psi}\sigma_{k0}\psi\,\dot{V}_k = -\frac{e^2}{16\pi^2 m}\overline{\psi}\sigma_{k0}\psi\,\dot{\tilde{p}}_k\,, \tag{749}$$

Using the definition $\sigma^{\rho\nu} = \frac{i}{2}[\gamma^\rho, \gamma^\nu]$, we find that this term is equal to

$$\frac{ie^2}{16\pi^2 m}\left(\psi^\dagger\boldsymbol{\gamma}\psi\right)\cdot\dot{\tilde{\mathbf{p}}} \tag{750}$$

If we continue analogously to the calculation of the contribution of the counter term, then the contribution from this vertex correction term towards the forward scattering amplitude, $\mathcal{F}_{\mathrm{Vtx}}$, can be written

$$\begin{aligned}\mathcal{F}_{\mathrm{Vtx}}(\mathbf{p}) &= -\frac{1}{\hbar}\int\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}\frac{m}{p_0}\int d^4x\langle 0|b_\alpha(\mathbf{p}')\left(\frac{ie^2}{16\pi^2 m}\left(\psi^\dagger(x)\boldsymbol{\gamma}\psi(x)\right)\cdot\dot{\tilde{\mathbf{p}}}\right)b^\dagger_\alpha(\mathbf{p})|0\rangle \\ &= -\frac{ie^2}{16\pi^2\hbar p_0}\int dt\,|\phi_p(t)|^2\left[u^\dagger_\alpha(p)\boldsymbol{\gamma}u_\alpha(p)\right]\cdot\dot{\tilde{\mathbf{p}}}\,.\end{aligned} \tag{751}$$

All momenta in the integrand are the time dependent $\tilde{p} = (E_p, \mathbf{p} - \mathbf{V}(t))$ and so we drop the tilde notation.[18] The spinor combination in the integrand of

[17]This interaction term is *part* of the forward-scattering already considered. We aim here to show that it is this part which is solely responsible for $\mathcal{F}^R$.

[18]This change of notation, performed for the purpose of simplicity and legibility, was also enacted in the main forward scattering calculation at a similar point and is thus also needed here for the purposes of comparison.

this equation can be straightforwardly calculated as follows:

$$\begin{aligned}
&\left[u_\alpha^\dagger(p)\boldsymbol{\gamma}u_\alpha(p)\right]\cdot\dot{\mathbf{p}} \\
&= \frac{E_p+m}{2m}\left(s_\alpha^\dagger \;\; s_\alpha^\dagger\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p+m}\right)\begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0\end{pmatrix}\begin{pmatrix} s_\alpha \\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p+m}s_\alpha\end{pmatrix}\dot{p}^k \\
&= \frac{1}{2m}s_\alpha^\dagger\left[\sigma^k,\sigma^j\right]s_\alpha p^j\dot{p}^k \\
&= \frac{i}{m}\epsilon^{nkj}s_\alpha^\dagger\sigma^n s_\alpha\, p^j\dot{p}^k \\
&= \frac{i}{m}\epsilon^{nkj}\xi^n p^j\dot{p}^k \\
&= -\frac{i}{m}\boldsymbol{\xi}\cdot(\mathbf{p}\times\dot{\mathbf{p}})\,. \qquad (752)
\end{aligned}$$

where ϵ^{nkj} is the usual Levi-Civita antisymmetric symbol. Substituting (752) in to (751) we find

$$\mathcal{F}_{\mathrm{Vtx}}(\mathbf{p}) = -\frac{e^2}{16\pi^2\hbar m p_0}\int dt\,|\phi_p(t)|^2\,\boldsymbol{\xi}\cdot(\mathbf{p}\times\dot{\mathbf{p}})\,. \qquad (753)$$

Comparison with (742) shows that

$$\mathcal{F}^R(\mathbf{p}) = \mathcal{F}_{\mathrm{Vtx}}(\mathbf{p})\,. \qquad (754)$$

We can consequently deduce that the correction to the forward scattering and thus the subsequent correction to the position shift are due to the renormalised one-loop vertex correction. Finally, we conclude that the quantum position shift for the spinor field in the $\hbar\to 0$ limit is given by

$$\delta x = -\int_{-\infty}^{0} dt\,\mathcal{F}_{\mathrm{LD}}^j\left(\frac{\partial x^j}{\partial p^i}\right)_t + \frac{e^2}{16\pi^2 m p_0}\partial_{p^i}\int dt\,|\phi_p(t)|^2\,\boldsymbol{\xi}\cdot(\mathbf{p}\times\dot{\mathbf{p}})\,. \qquad (755)$$

CHAPTER 7

Summary and Conclusion

> In this chapter we summarize the work which has been presented and discuss the results of our investigations. We also discuss possible avenues for future research on this topic.

In this work we have investigated the effects of radiation reaction in classical and quantum electrodynamics on the position of a particle. We defined the position shift to be the change in position due to the effects of radiation reaction and calculated this quantity for the theories with which we were interested. The equations of motion are a fundamental part of any theory, and the observation of dynamics is likewise fundamental to our ability to discern between rival theories and question our understanding. The reader may recall from the introduction, that the phenomenon of radiation reaction alters the usual equations of motion and consequently an understanding of radiation reaction and its effects lies at the heart of accurately understanding dynamics. One could regard the predicted position, or predicted position expectation value in order to fully include quantum theories, as one of the most important predictions of a theory. The change in this prediction after the addition of a new phenomenon, is consequently a sensible choice of measure to use in order to help understand our theoretical models.

The classical theory of radiation reaction is not without its problems, both in implementation and especially in interpretation. As we previously explained, most of these problems are related to the third order nature of the resulting equations of motion. It has been these difficulties, along with the recent renewed interest and progress on radiation reaction in curved space, that have motivated this work. Our aim has been to look at the effects of radiation reaction in classical electrodynamics and to compare the results with the

predictions of the so-called classical limit of the more fundamental quantum field theory. A knowledge of the similarities and differences between the two approaches to radiation reaction, and in turn the similarities and differences in the results of our investigations will hopefully aid a fuller understanding of how this phenomenon can be interpreted within our theoretical models. Given the debate about the classical theory and its interpretation, the natural question is whether the predictions of the quantum theory, in the classical limit, are the same as those of classical electrodynamics. This is one of the main questions that we sought to answer in this work.

Our model consisted of a particle interacting with an external potential for some finite period of time in the past of our measurement. The position shift was defined as the change in position between a hypothetic control particle which does not undergo radiation reaction, and a test particle which does include this effect. We refer the reader to the appropriate definitions of the models in the main chapters for the full description. The aim here is to recall these descriptions to mind. In classical electrodynamics, we treated the Lorentz-Dirac force, the classical radiation reaction force, as a perturbation. This is in keeping with the reduction in order interpretations of the theory, with the treatment of interactions in the perturbative description of quantum field theory, and with the fact that the Lorentz-Dirac force is a physically small effect. We demonstrated in Chapter 3 that the classical position shift can be given by[1]

$$\delta x^i_C = -\int_{-\infty}^{0} dt \mathcal{F}_{\mathrm{LD}}{}^j \left(\frac{\partial x^j}{\partial p^i}\right)_t . \tag{279}$$

This is a fairly short and simple expression and suggests a more general rule in addition to the case of the Lorentz-Dirac force. This is indeed the case

[1] The quantities and factors in the equations in this chapter are those defined in the main sections of this work. The equations numbers of the quoted results are the original equations numbers in the work.

and analysis of our working in Chapter 3 demonstrates that the above expression can be used for the position shift of a general perturbative force[2] to a Hamiltonian system within our model's set-up.[3]

Before tackling the quantum theories, we calculated the semiclassical expansions for the scalar and spinor field in Chapter 2. This work was necessary in order to later investigate the $\hbar \to 0$ limit of the quantum theories. The semiclassical expansions of the mode functions are dependent on the details of the acceleration due to the external potential and we calculated these expansions in the cases of the time-dependent (space-independent) potential, and only for the scalar field, in the case of the potential dependent on one of the spatial coordinates. These potentials were chosen due to the conditions for the validity of the semiclassical expansions. This chapter provided the ground work for our description of the quantum fields in the external potential.

Our first investigation into the quantum effects of radiation reaction used the theory of quantum scalar electrodynamics. The use of the scalar field is a good starting point for studying the quantum effects and a useful toy model for electrodynamics, without the complications of spin which is also absent in the classical model. We started our investigation with the calculation of the position expectation value of a non-radiating scalar particle, given by

$$\langle i|x^i(0)|i\rangle = \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}) \overleftrightarrow{\partial}_{p_i} f(\mathbf{p}) = 0 \quad \forall i = 1,2,3\,, \tag{287}$$

where we recall that f is heuristically to be regarded as the one-particle wave function. This then served as our control particle. We proceeded to calculate the position expectation value of a particle which has undergone radiation reaction during the period of acceleration and compared these two results. To order e^2, we found that there are two main processes contributing to the position shift. These are the emission and the forward scattering, which in

[2]For another force, $\mathcal{F}_{\mathrm{LD}}$ in (279) would of course need to be replaced by the equivalent expression for the new force.

[3]We again stress that our discussion here is within the limitations of the model we defined in full in Chapter 1.

turn come from the one photon and zero photon final states respectively:

$$\delta x^i_{\text{em}} = -\frac{i}{2}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^{\mu *}(\mathbf{p},\mathbf{k})\,\overleftrightarrow{\partial}_{p_i}\,\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\,, \tag{325}$$

$$\delta x^i_{\text{for}} = -\hbar\partial_{p_i}\Re\mathcal{F}(\mathbf{p})\,, \tag{324}$$

written in terms of the emission amplitude $\mathcal{A}$ and the forward scattering amplitude $\mathcal{F}$. The semiclassical expansions of the mode functions, described in Chapter 2 enabled us to calculate the amplitudes for these processes in the 'classical' $\hbar \to 0$ limit. For the emission amplitude we performed the calculation in the case of a time-dependent (space-independent) external potential and also in the case of a potential dependent on only one of the spatial coordinates. These two cases gave the same result, namely

$$\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) = -e\int_{-\infty}^{+\infty} d\xi\,\frac{dx^\mu}{d\xi}\chi(\xi)\,e^{ik\xi}\,, \tag{360}$$

Calculation of the resulting position shift due to the emission process produced

$$\delta x^i_{\text{em}} = -\int_{-\infty}^{0} dt\,\mathcal{F}^j_{\text{LD}}\left(\frac{\partial x^j}{\partial p^i}\right)_t\,. \tag{409}$$

In other words the position shift due to the emission process in the $\hbar \to 0$ limit of quantum scalar electrodynamics is equal to the classical position shift. Any difference between the classical and quantum measurements would thus need to arise from the forward scattering effects.

We thus proceeded to calculate the position shift due to forward scattering, for the case of the time-dependent potential. The forward scattering amplitude results from the one-loop interaction and we calculated the divergent contribution from these effects. However, this divergence was then subsequently found to be cancelled by the contribution from the counter term due to the renormalisation of the mass. These divergent expressions were both of order $\hbar^{-2}$ and thus, from the formula shown above in (324), they would contribute at order $\hbar^{-1}$ to the position shift. All remaining contributions from the forward scattering amplitude were shown to be imaginary at order $\hbar^{-1}$ in $\mathcal{F}$ and thus at order $\hbar^0$ in the position shift. The reader will recall that only the real part

of the forward scattering amplitude is present in the position shift formula. As a result the position shift contribution due to forward scattering is zero. We can consequently conclude that the quantum position shift for this model is equal to the classical position shift. This would at first sight appear to imply that there are no differences in the treatment of radiation reaction between the classical and quantum theories, at least within the confines of the models and in the $\hbar \to 0$ limit. However, the theoretical paths along which we travelled for these calculations have significant differences. In order to further expand on the similarities and differences, we turned to an alternative, but equivalent, description of the radiation reaction effect based on the Green's functions of the electromagnetic field (Chapter 5).

The key to the Green's function description of radiation reaction is the decomposition of the particle's retarded electromagnetic field into 'regular' and 'singular' components. The retarded Green's function, G_-, is decomposed into the regular and singular Green's functions, given respectively as

$$G_R = \frac{1}{2}\left[G_- - G_+\right] , \tag{38}$$

$$G_S = \frac{1}{2}\left[G_- + G_+\right] , \tag{37}$$

where G_+ is the advanced Green's function. The singular field is regarded as a generalisation of the Coulomb field for a static particle and is similarly singular (hence the name) on the world line of the particle. The regular Green's function, which solves the homogeneous wave equation, has been shown to be entirely responsible for the radiation reaction effect.[4] In Chapter 5 we showed that the emission contribution to the position shift can be rewritten as

$$\begin{aligned}\delta x_{\mathrm{em}} &= \int d^4x' d^4x''\, \partial_{p^i} j_{\rho''}(x) G_R^{\rho''\nu'}(x-x') j_{\nu'}(x') \\ &= \int d^4x\, \partial_{p^i} j^{\mu}(x) A_{\mathrm{R}\mu}(x)\, , \end{aligned}\tag{518}$$

[4]See, for example, [**18**].

i.e. in terms of the regular Green's function or regular field, A_R. As we demonstrated that the quantum position shift was entirely due to the emission amplitude contribution, the appearance of A_R sheds some light on the connection between the theories. The calculation leading to (518) started from the formula for the emission contribution to the position shift (325) and the emission amplitude obtained using the semiclassical expansion (360). We demonstrated in the scalar work that this amplitude result was obtained for a potential dependent on only one of the space-time coordinates. The position shift result will, however, hold for any potential that can be shown to produce the amplitude (360). The limitations of the semiclassical expansion dictated the use of the potentials mentioned, but given the form of the amplitude and its relation to the amplitude for a classical field, one would expect that the result may well be true for more general potentials, should an appropriate semiclassical method be applied. This possibility poses a question for future work.

Some major differences between the classical and quantum approaches come from the analysis of the forward scattering contribution, which however does still contain similarities. In the classical theory, the singular field is regarded as an infinite correction to the mass, and thus removed in a process of renormalisation. Thus in fact, the classical theory involves a divergent self-energy 'forward scattering' effect, removed by mass renormalisation, a process not normally associated with classical theories. Many people would think only of quantum theories when hearing the word renormalisation, but in both theories the mass renormalisation can be considering as arising from an infinite self-interaction effect. So far we have talked of the similarities. However, the quantum self-interaction as described by the one-loop process involves effects not present in the classical theory at all, such as contributions from the virtual antiparticles. In the calculation of the forward scattering in Chapter 4 we decomposed the forward scattering into particle and antiparticle loop contributions. The particle loop was further decomposed when we analysed the

low photon energy portion of the loop. It is this effect which is analogous to that present in the classical theory - the high photon energy limit and the antiparticle loop processes do not have classical counterparts. As this description hints, we found in Chapter 5 that the low photon energy particle loop contribution can be rewritten in terms of the singular field A_S to produce

$$-\hbar\partial_{p^i}\mathrm{Re}\,\mathcal{F}^{<}(\mathbf{p}) = \int d^4x\,\partial_{p^i}j^{\mu}(x)A_{S\mu}(x)\,, \tag{535}$$

which is an analogous expression to that for the emission contribution. We thus see that those elements of the quantum process with classical counterparts effectively give the same results as found in the classical theory. The total forward scattering contribution is zero after mass renormalisation and it is clear that the quantum mass renormalisation is not the same as the classical case, but naturally renormalises all the quantum contributions including the antiparticle loop.

The results in Chapter 5 do not change those of the previous chapter. They are simply a rewriting of some parts of the calculation in terms of different quantities. The results do however give a clearer picture of the similarities and differences between the classical and quantum treatment of radiation reaction.

Having succeeded in comparing the classical position shift with the scalar quantum position shift, we turned our attention to the more accurate quantum model of the spinor field. It is this field which is used in the standard theory of quantum electrodynamics and we thus repeated our investigation for the spinor QED. We investigated the case of the spinor wave packet having travelled through a time-dependent potential. The position expectation value of the control particle was found, to order $\hbar^0$, to give the same expression as previously found in the case of the scalar field:

$$\langle x^i\rangle|_{t=0} = \frac{i\hbar}{2}\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}f^*(\mathbf{p})\,\overleftrightarrow{\partial}_{p_i}\,f(\mathbf{p})\,. \tag{550}$$

However, as an aside, we did note that spin effects, related to spin-orbit coupling, can be observed to have an effect at order $\hbar$. Such spin effects would still not however alter our measured position shift due to their presence in both

the control and test particle calculations. The addition of the phenomenon of spin to the model is one major difference between the classical electrodynamics theory and spinor QED, although it should be noted that the spin is present in the Dirac equation for the spinor field and is not technically a result of the quantisation of that field. Proceeding, we found that as per the scalar field, the contributions to the position shift can be split into emission and forward scattering contributions. In fact, the formula for the position shift, written in terms of the amplitudes for these processes was calculated to give the same answer as the scalar field:

$$\delta x^i_{\text{em}} = -\frac{i}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0}\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial}_{p_i} \mathcal{A}^\mu(\mathbf{p},\mathbf{k}) \tag{572}$$

$$\delta x^i_{\text{for}} = -\hbar\partial_{p_i}\Re\mathcal{F}(\mathbf{p})\,. \tag{573}$$

Again, the effects of the spin, including any spin transport effects producing a difference between the initial and final spin states, did not come into play at lowest order.

Despite the more complicated nature of the Dirac spinor field in comparison with the scalar field, the interaction Hamiltonian for the spinor field

$$\mathcal{H}_I = e :\bar{\psi}\not{A}\psi :\,, \tag{116}$$

is simpler than the scalar case. We used the spinor interactions and semiclassical spinor solutions to proceed to calculate the emission and forward scattering amplitudes. Many of the features of these calculations were analogous to those found in the previous scalar work. In fact, the result of the emission amplitude calculation in the $\hbar \to 0$ limit gave the same result as the emission amplitude for the scalar field, viz

$$\mathcal{A}^\mu(p,k) = -e\int d\xi\frac{dx^\mu}{d\xi}\chi(\xi)e^{ik\xi}\,. \tag{756}$$

Using either the direct calculation in Chapter 4 or the Green's function decomposition method in Chapter 5, we arrive at the result that the emission contribution to the quantum position shift for the spinor field is equal to the

classical position shift.[5] So far the result of changing field has merely been to change the intermediate calculations, rather than the final result. The pattern of the calculation for the forward scattering amplitude for the spinor field was initially similar to that for the scalar case. For example, the particle and antiparticle loop contributions were calculated and for the case of the particle loop it was again necessary to check the $\hbar$ order of the infrared divergences and analyse the low-energy contribution. The renormalisation of the mass via the counterterm again removed the order $\hbar^{-1}$ contributions to the position shift. However, unlike the scalar case, we had to additionally consider order $\hbar$ terms in the semiclassical expansion. As the forward scattering amplitude is at order $\hbar^{-2}$ and thus its contribution to the position shift at order $\hbar^{-1}$, any order $\hbar$ effects in the expansion potentially contribute at order $\hbar^0$ to the position shift and consequently remain when the classical limit is taken. After renormalisation, some of these terms in the amplitude were indeed still present and after some integration and simplification we were able to show that the 'correction' to the forward scattering amplitude is given by

$$\mathcal{F}^R(\mathbf{p}) = -\frac{e^2}{16\pi^2 p_0 m\hbar} \int dt |\phi_p(t)|^2 \boldsymbol{\xi} \cdot \mathbf{p} \times \dot{\mathbf{p}}\,. \tag{742}$$

This result then gives an additional contribution to the position shift when compared with either the quantum scalar case or the classical result. The position shift for the spinor quantum position shift was given at the end of Chapter 6 by

$$\delta x = -\int_{-\infty}^{0} dt\, \mathcal{F}^j_{\mathrm{LD}} \left(\frac{\partial x^j}{\partial p^i}\right)_t + \frac{e^2}{16\pi^2 m p_0} \partial_{p^i} \int dt\, |\phi_p(t)|^2 \boldsymbol{\xi} \cdot (\mathbf{p} \times \dot{\mathbf{p}})\,. \tag{755}$$

The interpretation of this extra term was analysed in the last subsection of that chapter. We found that the correction was entirely the result of the renormalised vertex correction that is produced by the one-loop process. In other words it is the result of the correction to the coupling to the external field produced at the one-loop level. As we described in our discussion at the end of

[5]Throughout this summary, we imply the $\hbar \to 0$ limit when talking about the quantum position shift results.

Chapter 6, the vertex correction is responsible for the well known anomalous magnetic moment. This correction is not simply a result of the addition of spin. The spin is present in the Dirac equation prior to quantisation and produces the prediction that the g-factor of the magnetic moment[6] is equal to 2. The anomalous magnetic moment is the correction to the g-factor due to quantisation, starting at the one-loop level. The effect to the position shift noted above is of similar origin.

Given the correction produced by quantisation the natural question to ask is whether or not this effect can be measured. This is of course an interesting question, and indeed any measurements improving the accuracy of tests of radiation reaction would be beneficial to our understanding of the phenomenon. The smallness of the radiation reaction force was stated as one of the reasons that it was frequently ignored and it is also one which hampers accurate testing of the theories. It is also worth noting however, that the purpose of the work presented here was to study the classical limit of QED. This naturally leads us to consider further work and the possibility of analysing the effect at higher orders in $\hbar$. Indeed Higuchi and Walker are currently investigating the $\hbar$ correction to the Larmor formula [**37**], which would have some influence on such an extension to this work. In addition, spin effects can be shown to be orders of magnitude larger than the self-force at low-energies (see for example [**38**]) and consequently they should be considered when predictions for possible experiments are made. Such investigations thus present a natural extension for future investigation and would aid the understanding of the current results by adding additional context. They would also require further investigation into the semiclassical expansion at higher orders, or an alternative method for such expansions to include other more general external potentials.

Additional directions in which this work can be extended include the possibility of investigating quantum radiation reaction in a curved space setting.

[6]The magnetic moment due to the intrinsic angluar momentum from the spin $\mathbf{s}$ is given by $\mu = -g\,e\,\mathbf{s}/(2m)$.

In the introduction in Chapter 1, we presented a brief summary of the theory of radiation reaction in curved space and noted that this is an area of great current interest. Much of the interest is focused on the effects of gravitational radiation reaction. It would be of great interest to extend the current work to consider quantum electromagnetic radiation reaction in curved space. In curved space this could be linked to investigations of the radiation produced by the expansion of space-time (see for example [**39**]). Further work could then attempt to grapple with a quantum treatment of gravitational radiation and gravitational radiation reaction. Due to the fact that the self-force is fundamental to our full understanding of dynamics and even on a classical level involves many of the concepts which usually define the complications of quantum theories, such as self-interaction and renormalisation, it may provide a useful avenue in which to obtain further knowledge of quantum fields in curved space and ultimately, signals towards the ever elusive theory of quantum gravity. For the purposes of working in curved space, the Green's function decomposition approach may well be more suited to adaptation for curved space given the methods used in both classical radiation reaction in curved space[7] and also in the treatment of quantum fields in curved space.[8]

The main focus of future work is therefore to build on the work presented here, using it as a base upon which to generalise the results presented. The generalisations mentioned above and in the main text include extensions to higher orders in $\hbar$, extensions to more general external potentials and extensions to curved space and radiation reaction in other fields. The work presented here has provided a solid base for future investigation and has given us new insight into the similarities and differences between the classical and quantum treatments of radiation reaction. The author hopes that the reader has found this report to be interesting, to answer the some of questions posed about

[7]See the earlier introduction and the much more detailed review by Poisson in [**19**].

[8]See, for example, [**35**].

radiation reaction, and perhaps to advance further questions in the reader's mind to be answered in future.

APPENDIX A

Semiclassical Spinor Identities

In this appendix we derive a set of identities for combinations of the time-dependent semiclassical spinors, expanded up to $\mathcal{O}(\hbar)$. These identities can then be used, for example, in the evaluation of the spinor combinations in the forward scattering loops.

1. Summary of semiclassical expansions

Firstly we quote the semiclassical spinors derived in the semiclassical chapter. For the particle spinors u, all energy-momenta are $\tilde{p} = (E_p, \tilde{\mathbf{p}})$ where

$$\tilde{\mathbf{p}} = \mathbf{p} - \mathbf{V}(t) \qquad\qquad E_p = \sqrt{\tilde{\mathbf{p}}^2 + m^2}\,, \tag{757}$$

and for the antiparticle spinors v, we have $\tilde{p}_+ = (E_{p+}, \tilde{\mathbf{p}}_+)$ where

$$\tilde{\mathbf{p}}_+ = \mathbf{p} + \mathbf{V}(t) \qquad\qquad E_p = \sqrt{\tilde{\mathbf{p}}_+^2 + m^2}\,. \tag{758}$$

With this in mind, when there is no ambiguity, we drop the momenta subscripts. We recall

$$u_\alpha(p,t) = \sqrt{\frac{E+m}{2m}}\left[(1+i\hbar g)\begin{pmatrix} U s_\alpha \\ \Sigma U s_\alpha \end{pmatrix} - i\hbar\frac{E+m}{(2E_p)^2}\begin{pmatrix} -\Sigma\dot{\Sigma} U s_\alpha \\ \dot{\Sigma} U s_\alpha \end{pmatrix}\right]\,, \tag{759}$$

$$\begin{aligned}\bar{u}_\alpha(p,t) = \sqrt{\frac{E+m}{2m}}\Big[&(1-i\hbar g)\begin{pmatrix} s_\alpha^\dagger U^\dagger & -s_\alpha^\dagger U^\dagger \Sigma \end{pmatrix} \\ &+ i\hbar\frac{E+m}{(2E_p)^2}\begin{pmatrix} -s_\alpha^\dagger U^\dagger \dot{\Sigma}\Sigma & -s_\alpha^\dagger U^\dagger \dot{\Sigma} \end{pmatrix}\Big]\,,\end{aligned} \tag{760}$$

$$v_\alpha(p,t) = \sqrt{\frac{E+m}{2m}}\left[(1-i\hbar g)\begin{pmatrix} \Sigma U s_\alpha \\ U s_\alpha \end{pmatrix} + i\hbar\frac{E+m}{(2E_p)^2}\begin{pmatrix} \dot{\Sigma} U s_\alpha \\ -\Sigma\dot{\Sigma} U s_\alpha \end{pmatrix}\right]\,, \tag{761}$$

$$\bar{v}_\alpha(p,t) = \sqrt{\frac{E+m}{2m}}\Big[(1+i\hbar g)\left(s_\alpha^\dagger U^\dagger\Sigma \quad -s_\alpha^\dagger U^\dagger\right) \\ -i\hbar\frac{E+m}{(2E_p)^2}\left(s_\alpha^\dagger U^\dagger\dot{\Sigma} \quad s_\alpha^\dagger U^\dagger\dot{\Sigma}\Sigma\right)\Big]\,, \tag{762}$$

with the following (full) notation:

$$\Lambda_p(t) = \frac{\boldsymbol{\sigma}\cdot\tilde{\mathbf{p}}(t)\times\dot{\tilde{\mathbf{p}}}}{E_p+m}\,, \tag{763}$$

$$U_p(t) = T\left(\exp\left[-i\int_0^t d\tau\frac{\Lambda_p(\tau)}{2E_p(\tau)}\right]\right)\,, \tag{764}$$

$$\Sigma_p(t) = \frac{\boldsymbol{\sigma}\cdot\tilde{\mathbf{p}}}{E_p+m}\,, \tag{765}$$

$$\dot{\Sigma}_p(t) = \frac{\boldsymbol{\sigma}}{E_p+m}\cdot\left(\dot{\tilde{\mathbf{p}}}(t) - \frac{\dot{E}_p)\tilde{\mathbf{p}}}{E_p+m}\right)\,, \tag{766}$$

$$g_p(t) = \int_0^t d\tau\frac{\dot{\tilde{p}}^2(\tau)}{8E_p^3(\tau)}\,, \tag{767}$$

$$\tag{768}$$

and similarly for $\tilde{p}_+$. We note that Λ, Σ and $\dot{\Sigma}$ are Hermitian, whilst U is unitary.

2. Summary of useful identities

The following are identities involving some of the terms above which are useful for the calculation of the spinor identities. For the purpose of the summary we use the time-dependent energy momentum $(E,\mathbf{p})$.

$$\Sigma^2 = \frac{E-m}{E+m}\,, \tag{769}$$

$$\Sigma\dot{\Sigma} = \frac{m\dot{E}}{(E+m)^2} + \frac{i\boldsymbol{\sigma}\cdot\mathbf{p}\times\dot{\mathbf{p}}}{(E+m)^2}\,, \tag{770}$$

$$\Sigma\dot{\Sigma}\Sigma = \frac{\dot{E}\boldsymbol{\sigma}\cdot\mathbf{p}}{(E+m)^2} - \frac{(E-m)\,\boldsymbol{\sigma}\cdot\dot{\mathbf{p}}}{(E+m)^2}\,, \tag{771}$$

$$\dot{\Sigma}+\Sigma\dot{\Sigma}\Sigma = \frac{2m\boldsymbol{\sigma}\cdot\dot{\mathbf{p}}}{(E+m)^2}\,, \tag{772}$$

$$\Sigma\boldsymbol{\sigma} = \frac{\mathbf{p}}{E+m} + \frac{i\boldsymbol{\sigma}\times\mathbf{p}}{E+m}\,, \tag{773}$$

$$\boldsymbol{\sigma}\Sigma = \frac{\mathbf{p}}{E+m} - \frac{i\boldsymbol{\sigma}\times\mathbf{p}}{E+m}\,, \tag{774}$$

$$\Sigma\boldsymbol{\sigma}\Sigma = \frac{2\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}}{(E+m)^2} - \frac{\mathbf{p}^2\boldsymbol{\sigma}}{(E+m)^2}\,, \tag{775}$$

$$\dot{\Sigma}\Sigma\boldsymbol{\sigma}\Sigma = \frac{\mathbf{p}\cdot\dot{\mathbf{p}}\mathbf{p} - \mathbf{p}^2\dot{\mathbf{p}}}{(E+m)^3} - \frac{2i\boldsymbol{\sigma}\cdot\mathbf{p}\times\dot{\mathbf{p}}\mathbf{p}}{(E+m)^3} - \frac{i\mathbf{p}^2}{(E+m)^3}\boldsymbol{\sigma}\times\left(\dot{\mathbf{p}} - \frac{\dot{E}\mathbf{p}}{E+m}\right). \tag{776}$$

3. Zeroth order spinor identities

Below are the standard zeroth order spinor identities showing the normalisation we have used for the spinors:

$$\bar{u}^{(0)}_\alpha(p)u^{(0)}_\alpha(p) = 1\,, \tag{777}$$

$$\bar{u}^{(0)}_\alpha(p)\gamma^0u^{(0)}_\alpha(p) = \frac{E}{m}\,, \tag{778}$$

$$\bar{u}^{(0)}_\alpha(p)\boldsymbol{\gamma}u^{(0)}_\alpha(p) = \frac{\mathbf{p}}{m}\,, \tag{779}$$

$$u^{(0)}_\alpha(p)\bar{u}^{\alpha(0)}(p) = \frac{\gamma\cdot p+m}{2m}\,. \tag{780}$$

$$\bar{v}^{(0)}_\alpha(p)v^{(0)}_\alpha(p) = -1\,, \tag{781}$$

$$\bar{v}^{(0)}_\alpha(p)\gamma^0v^{(0)}_\alpha(p) = \frac{E}{m}\,, \tag{782}$$

$$\bar{v}^{(0)}_\alpha(p)\boldsymbol{\gamma}v^{(0)}_\alpha(p) = \frac{\mathbf{p}}{m}\,, \tag{783}$$

$$v^{(0)}_\alpha(p)\bar{v}^{\alpha(0)}(p) = \frac{\gamma\cdot p-m}{2m}\,. \tag{784}$$

4. Equal time spinor identities

Here we present the identities to $\mathcal{O}(\hbar)$ for both the particle ('positive energy') and antiparticle ('negative energy') semiclassical spinors in turn, evaluated with the same (time-dependent) momenta at equal time. The identities will be useful in the calculation of the outer spinors in the combinations found in the forward scattering contributions. All momenta are given by p at, say,

time $\bar{t}$. Consequently for simplicity of notation we shall drop the explicit p subscripts and time arguments. Similarly we can treat $s_\alpha(\bar{t}) = s_\alpha U_p(\bar{t})$ as the two-spinor for the direction of the spin at that time and, as all times are equal, drop the argument notation.

4.1. Particle equal time spinors. The first set of identities are for the particle spinors.

4.1.1.

$$
\begin{aligned}
&\bar{u}_\alpha(p)u_\alpha(p) \\
&= \frac{E+m}{2m}\Bigg[(1-i\hbar g)\begin{pmatrix} s_\alpha^\dagger & -s_\alpha^\dagger\Sigma \end{pmatrix} \\
&\qquad\qquad \times\left((1+i\hbar g)\begin{pmatrix} s_\alpha \\ \Sigma s_\alpha \end{pmatrix} - i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix} -\Sigma\dot{\Sigma}s_\alpha \\ \dot{\Sigma}s_\alpha \end{pmatrix}\right) \\
&\qquad\quad + i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix} -s_\alpha^\dagger\dot{\Sigma}\Sigma & -s_\alpha^\dagger\dot{\Sigma} \end{pmatrix}\begin{pmatrix} s_\alpha \\ \Sigma s_\alpha \end{pmatrix}\Bigg] \\
&= \frac{E+m}{2m}\left[s_\alpha^\dagger\left(1-\Sigma^2\right)s_\alpha - i\hbar\frac{E+m}{(2E)^2}s_\alpha^\dagger\left(-\Sigma\dot{\Sigma} - \Sigma\dot{\Sigma} + \dot{\Sigma}\Sigma + \dot{\Sigma}\Sigma\right)s_\alpha\right] \\
&= \frac{E+m}{2m}\left[\frac{2m}{E+m} - i\hbar\frac{E+m}{2E^2}s_\alpha^\dagger\left(-\frac{2i\Lambda}{E+m}\right)s_\alpha\right] ,
\end{aligned}
$$

which leads to

$$
\bar{u}_\alpha(p)u_\alpha(p) = 1 - \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\mathbf{p}\times\dot{\mathbf{p}}}{E^2} \,. \tag{785}
$$

4.1.2.

$$
\begin{aligned}
&\bar{u}_\alpha(p)\gamma^0 u_\alpha(p) \\
&= \frac{E+m}{2m}\left[(1-i\hbar g)\begin{pmatrix} s_\alpha^\dagger & s_\alpha^\dagger \Sigma \end{pmatrix}\right. \\
&\qquad\qquad \times \left((1+i\hbar g)\begin{pmatrix} s_\alpha \\ \Sigma s_\alpha \end{pmatrix} - i\hbar \frac{E+m}{(2E)^2}\begin{pmatrix} -\Sigma\dot{\Sigma} s_\alpha \\ \dot{\Sigma} s_\alpha \end{pmatrix}\right) \\
&\qquad \left. + i\hbar \frac{E+m}{(2E)^2}\begin{pmatrix} -s_\alpha^\dagger \dot{\Sigma}\Sigma & s_\alpha^\dagger \dot{\Sigma} \end{pmatrix}\begin{pmatrix} s_\alpha \\ \Sigma s_\alpha \end{pmatrix}\right] \\
&= \frac{E+m}{2m}\left[s_\alpha^\dagger\left(1+\Sigma^2\right)s_\alpha - i\hbar\frac{E+m}{(2E)^2} s_\alpha^\dagger\left(-\Sigma\dot{\Sigma} + \Sigma\dot{\Sigma} + \dot{\Sigma}\Sigma - \dot{\Sigma}\Sigma\right)s_\alpha\right] \\
&= \frac{E+m}{2m}\frac{2E}{E+m}\,,
\end{aligned}
$$

and hence we have

$$
\bar{u}_\alpha(p)\gamma^0 u_\alpha(p) = \frac{E}{m}\,. \tag{786}
$$

We note that there is no order $\hbar$ term remaining.

4.1.3.

$$
\begin{aligned}
&\bar{u}_\alpha(p)\boldsymbol{\gamma} u_\alpha(p) \\
&= \frac{E+m}{2m}\left[(1-i\hbar g)\begin{pmatrix} s_\alpha^\dagger \Sigma\boldsymbol{\sigma} & s_\alpha^\dagger \boldsymbol{\sigma} \end{pmatrix}\right. \\
&\qquad\qquad \times \left((1+i\hbar g)\begin{pmatrix} s_\alpha \\ \Sigma s_\alpha \end{pmatrix} - i\hbar \frac{E+m}{(2E)^2}\begin{pmatrix} -\Sigma\dot{\Sigma} s_\alpha \\ \dot{\Sigma} s_\alpha \end{pmatrix}\right) \\
&\qquad \left. + i\hbar \frac{E+m}{(2E)^2}\begin{pmatrix} s_\alpha^\dagger \dot{\Sigma}\boldsymbol{\sigma} & -s_\alpha^\dagger \dot{\Sigma}\Sigma\boldsymbol{\sigma} \end{pmatrix}\begin{pmatrix} s_\alpha \\ \Sigma s_\alpha \end{pmatrix}\right]\,.
\end{aligned}
$$

Multiplying the matrices, we have

$$
\begin{aligned}
\bar{u}_\alpha(p)\boldsymbol{\gamma} u_\alpha(p) = \frac{E+m}{2m}&\left[s_\alpha^\dagger\left(\Sigma\boldsymbol{\sigma} + \boldsymbol{\sigma}\Sigma\right)s_\alpha\right. \\
&\left. - i\hbar\frac{E+m}{(2E)^2} s_\alpha^\dagger\left(\dot{\Sigma}\Sigma\boldsymbol{\sigma}\Sigma - \Sigma\boldsymbol{\sigma}\Sigma\dot{\Sigma} + \boldsymbol{\sigma}\dot{\Sigma} - \dot{\Sigma}\boldsymbol{\sigma}\right)s_\alpha\right]\,,
\end{aligned}
$$

which using the identities in section 2 becomes

$$= \frac{E+m}{2m}\left[\frac{2\mathbf{p}}{E+m} - i\hbar\frac{E+m}{(2E)^2}s_\alpha^\dagger\left(-\frac{4i\boldsymbol{\sigma}\cdot\mathbf{p}\times\dot{\mathbf{p}}\mathbf{p}}{(E+m)^3}\right.\right.$$
$$\left.\left. - \left(\frac{2i\mathbf{p}^2}{(E+m)^3} + \frac{2i}{E+m}\right)\boldsymbol{\sigma}\times\left(\dot{\mathbf{p}} - \frac{\dot{E}\mathbf{p}}{E+m}\right)\right)s_\alpha\right]. \quad (787)$$

Thus

$$\bar{u}_\alpha(p)\boldsymbol{\gamma}u_\alpha(p) = \frac{\mathbf{p}}{m} - \frac{\hbar}{2mE}\left(\frac{\boldsymbol{\xi}\cdot\mathbf{p}\times\dot{\mathbf{p}}\,\mathbf{p}}{E(E+m)} + \boldsymbol{\xi}\times\left(\dot{\mathbf{p}} - \frac{\dot{E}\mathbf{p}}{E+m}\right)\right). \quad (788)$$

4.1.4. *Inner spinors.* The last identity for the ‘particle’ spinors we present here is used for the inner spinors at equal time.

$$u_\alpha(p)\bar{u}^\alpha(p)$$
$$= \sum_\alpha \frac{E+m}{2m}\left[(1+i\hbar g)\begin{pmatrix} s_\alpha \\ \Sigma s_\alpha\end{pmatrix}\right.$$
$$\times\left[(1-i\hbar g)\begin{pmatrix} s_\alpha^\dagger & -s_\alpha^\dagger\Sigma\end{pmatrix} + i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix} -s_\alpha^\dagger\dot{\Sigma}\Sigma & -s_\alpha^\dagger\dot{\Sigma}\end{pmatrix}\right]$$
$$\left. -i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix} -\Sigma\dot{\Sigma}s_\alpha \\ \dot{\Sigma}s_\alpha\end{pmatrix}\begin{pmatrix} s_\alpha^\dagger & -s_\alpha^\dagger\Sigma\end{pmatrix}\right]$$
$$= \sum_\alpha \frac{E+m}{2m}\left[\begin{pmatrix} s_\alpha s_\alpha^\dagger & -s_\alpha s_\alpha^\dagger\Sigma \\ \Sigma s_\alpha s_\alpha^\dagger & -\Sigma s_\alpha s_\alpha^\dagger\Sigma\end{pmatrix} + i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix} -s_\alpha s_\alpha^\dagger\dot{\Sigma}\Sigma & -s_\alpha s_\alpha^\dagger\dot{\Sigma} \\ -\Sigma s_\alpha s_\alpha^\dagger\dot{\Sigma}\Sigma & -\Sigma s_\alpha s_\alpha^\dagger\dot{\Sigma}\end{pmatrix}\right.$$
$$\left. -i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix} -\Sigma\dot{\Sigma}s_\alpha s_\alpha^\dagger & \Sigma\dot{\Sigma}s_\alpha s_\alpha^\dagger\Sigma \\ \dot{\Sigma}s_\alpha s_\alpha^\dagger & -\dot{\Sigma}s_\alpha s_\alpha^\dagger\Sigma\end{pmatrix}\right]$$
$$= \frac{E+m}{2m}\left[\begin{pmatrix} I & -\Sigma \\ \Sigma & -\Sigma^2\end{pmatrix} - i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix} \dot{\Sigma}\Sigma - \Sigma\dot{\Sigma} & \dot{\Sigma} + \Sigma\dot{\Sigma}\Sigma \\ \dot{\Sigma} + \Sigma\dot{\Sigma}\Sigma & \Sigma\dot{\Sigma} - \dot{\Sigma}\Sigma\end{pmatrix}\right]$$
$$= \frac{1}{2m}\left[\begin{pmatrix} E+m & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -E+m\end{pmatrix} - i\hbar\frac{1}{(2E)^2}\begin{pmatrix} -2i\boldsymbol{\sigma}\cdot\mathbf{p}\times\dot{\mathbf{p}} & 2m\boldsymbol{\sigma}\cdot\dot{\mathbf{p}} \\ 2m\boldsymbol{\sigma}\cdot\dot{\mathbf{p}} & 2i\boldsymbol{\sigma}\cdot\mathbf{p}\times\dot{\mathbf{p}}\end{pmatrix}\right].$$

Using the gamma matrices, we can rewrite this equation as

$$u_\alpha(p)\bar{u}^\alpha(p) = \frac{\gamma\cdot p + m}{2m} + \frac{\hbar}{m(2E)^2}\gamma^5\boldsymbol{\gamma}\cdot\mathbf{p}\times\dot{\mathbf{p}} - \frac{i\hbar}{(2E)^2}\gamma^0\boldsymbol{\gamma}\cdot\dot{\mathbf{p}}. \quad (789)$$

4.2. Antiparticle equal time spinors. We now repeat these identities for the antiparticle spinors.

4.2.1.

$$
\begin{aligned}
&\bar{v}_\alpha(p)v_\alpha(p)\\
&= \frac{E+m}{2m}\Bigg[(1+i\hbar g)\begin{pmatrix} s_\alpha^\dagger\Sigma & -s_\alpha^\dagger\end{pmatrix}\\
&\qquad\times\left((1-i\hbar g)\begin{pmatrix}\Sigma s_\alpha\\ s_\alpha\end{pmatrix} + i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix}\dot{\Sigma}s_\alpha\\ -\Sigma\dot{\Sigma}s_\alpha\end{pmatrix}\right)\\
&\qquad -i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix} s_\alpha^\dagger\dot{\Sigma} & s_\alpha^\dagger\dot{\Sigma}\Sigma\end{pmatrix}\begin{pmatrix}\Sigma s_\alpha\\ s_\alpha\end{pmatrix}\Bigg]\\
&= \frac{E+m}{2m}\left[s_\alpha^\dagger\left(\Sigma^2-1\right)s_\alpha + i\hbar\frac{E+m}{2E^2}s_\alpha^\dagger\left(\Sigma\dot{\Sigma}-\dot{\Sigma}\Sigma\right)s_\alpha\right]\\
&= \frac{E+m}{2m}\left[-\frac{2m}{E+m} + i\hbar\frac{E+m}{2E^2}s_\alpha^\dagger\frac{2i\Lambda}{E+m}s_\alpha\right].
\end{aligned}
$$

Thus we obtain

$$
\bar{v}_\alpha(p)v_\alpha(p) = -1 - \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\mathbf{p}\times\dot{\mathbf{p}}}{E^2}. \tag{790}
$$

4.2.2.

$$
\begin{aligned}
&\bar{v}_\alpha(p)\gamma^0 v_\alpha(p)\\
&= \frac{E+m}{2m}\Bigg[(1+i\hbar g)\begin{pmatrix} s_\alpha^\dagger\Sigma & s_\alpha^\dagger\end{pmatrix}\\
&\qquad\times\left((1-i\hbar g)\begin{pmatrix}\Sigma s_\alpha\\ s_\alpha\end{pmatrix} + i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix}\dot{\Sigma}s_\alpha\\ -\Sigma\dot{\Sigma}s_\alpha\end{pmatrix}\right)\\
&\qquad -i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix} s_\alpha^\dagger\dot{\Sigma} & -s_\alpha^\dagger\dot{\Sigma}\Sigma\end{pmatrix}\begin{pmatrix}\Sigma s_\alpha\\ s_\alpha\end{pmatrix}\Bigg]\\
&= \frac{E+m}{2m}\left[s_\alpha^\dagger\left(\Sigma^2+1\right)s_\alpha + i\hbar\frac{E+m}{(2E)^2}s_\alpha^\dagger\left(\Sigma\dot{\Sigma}-\Sigma\dot{\Sigma}-\dot{\Sigma}\Sigma+\dot{\Sigma}\Sigma\right)s_\alpha\right]\\
&= \frac{E+m}{2m}\frac{2E}{E+m}.
\end{aligned}
$$

We therefore once again have

$$\bar{v}_\alpha(p)\gamma^0 v_\alpha(p) = \frac{E}{m}\,, \tag{791}$$

with no order $\hbar$ term.

4.2.3.

$$\begin{aligned}
&\bar{v}_\alpha(p)\boldsymbol{\gamma} v_\alpha(p) \\
&= \frac{E+m}{2m}\Big[(1+i\hbar g)\begin{pmatrix} s_\alpha^\dagger \boldsymbol{\sigma} & s_\alpha^\dagger \Sigma \boldsymbol{\sigma}\end{pmatrix} \\
&\qquad\qquad \times \left((1-i\hbar g)\begin{pmatrix}\Sigma s_\alpha \\ s_\alpha\end{pmatrix} + i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix}\dot{\Sigma} s_\alpha \\ -\Sigma\dot{\Sigma} s_\alpha\end{pmatrix}\right) \\
&\qquad\quad -i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix} -s_\alpha^\dagger \dot{\Sigma}\Sigma\boldsymbol{\sigma} & s_\alpha^\dagger \dot{\Sigma}\boldsymbol{\sigma}\end{pmatrix}\begin{pmatrix}\Sigma s_\alpha \\ s_\alpha\end{pmatrix}\Big]\,,
\end{aligned}$$

which gives

$$\begin{aligned}
\bar{v}_\alpha(p)\boldsymbol{\gamma} v_\alpha(p) = \frac{E+m}{2m}\Big[& s_\alpha^\dagger\left(\boldsymbol{\sigma}\Sigma + \Sigma\boldsymbol{\sigma}\right)s_\alpha \\
& + i\hbar\frac{E+m}{(2E)^2} s_\alpha^\dagger\left(\boldsymbol{\sigma}\dot{\Sigma} - \dot{\Sigma}\boldsymbol{\sigma} + \dot{\Sigma}\Sigma\boldsymbol{\sigma}\Sigma - \Sigma\boldsymbol{\sigma}\Sigma\dot{\Sigma}\right)s_\alpha\Big]\,.
\end{aligned}$$

Using the identities presented in section 2, we find

$$\begin{aligned}
\bar{v}_\alpha(p)\boldsymbol{\gamma} v_\alpha(p) = \frac{E+m}{2m}\Bigg[& \frac{2\mathbf{p}}{E+m} + i\hbar\frac{E+m}{(2E)^2} s_\alpha^\dagger\Bigg[-\frac{4i\boldsymbol{\sigma}\cdot\mathbf{p}\times\dot{\mathbf{p}}\mathbf{p}}{(E+m)^3} \\
& -\left(\frac{2i\mathbf{p}^2}{(E+m)^3} + \frac{2i}{E+m}\right)\boldsymbol{\sigma}\times\left(\dot{\mathbf{p}} - \frac{\dot{E}\mathbf{p}}{E+m}\right)\Bigg]s_\alpha\Bigg]\,,
\end{aligned}$$

and so

$$\bar{v}_\alpha(p)\boldsymbol{\gamma} v_\alpha(p) = \frac{\mathbf{p}}{m} + \frac{\hbar}{2mE}\left(\frac{\boldsymbol{\xi}\cdot\mathbf{p}\times\dot{\mathbf{p}}\,\mathbf{p}}{E(E+m)} + \boldsymbol{\xi}\times\left(\dot{\mathbf{p}} - \frac{\dot{E}\mathbf{p}}{E+m}\right)\right)\,. \tag{792}$$

4.2.4. *Inner spinors.*

$$
\begin{aligned}
&v_\alpha(p)\bar{v}^\alpha(p)\\
&= \sum_\alpha \frac{E+m}{2m}\left[(1-i\hbar g)\begin{pmatrix}\Sigma s_\alpha\\ s_\alpha\end{pmatrix}\right.\\
&\qquad \times\left[(1+i\hbar g)\begin{pmatrix}s_\alpha^\dagger\Sigma & -s_\alpha^\dagger\end{pmatrix} - i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix}s_\alpha^\dagger\dot{\Sigma} & s_\alpha^\dagger\dot{\Sigma}\Sigma\end{pmatrix}\right]\\
&\qquad \left. + i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix}\dot{\Sigma}s_\alpha\\ -\Sigma\dot{\Sigma}s_\alpha\end{pmatrix}\begin{pmatrix}s_\alpha^\dagger\Sigma & -s_\alpha^\dagger\end{pmatrix}\right]\\
&= \sum_\alpha \frac{E+m}{2m}\left[\begin{pmatrix}\Sigma s_\alpha s_\alpha^\dagger\Sigma & -\Sigma s_\alpha s_\alpha^\dagger\\ s_\alpha s_\alpha^\dagger\Sigma & -s_\alpha s_\alpha^\dagger\end{pmatrix} - i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix}\Sigma s_\alpha s_\alpha^\dagger\dot{\Sigma} & \Sigma s_\alpha s_\alpha^\dagger\dot{\Sigma}\Sigma\\ s_\alpha s_\alpha^\dagger\dot{\Sigma} & s_\alpha s_\alpha^\dagger\dot{\Sigma}\Sigma\end{pmatrix}\right.\\
&\qquad \left. + i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix}\dot{\Sigma}s_\alpha s_\alpha^\dagger\Sigma & -\dot{\Sigma}s_\alpha s_\alpha^\dagger\\ -\Sigma\dot{\Sigma}s_\alpha s_\alpha^\dagger\Sigma & \Sigma\dot{\Sigma}s_\alpha s_\alpha^\dagger\end{pmatrix}\right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&v_\alpha(p)\bar{v}^\alpha(p)\\
&= \frac{E+m}{2m}\left[\begin{pmatrix}\Sigma^2 & -\Sigma\\ \Sigma & -I\end{pmatrix} - i\hbar\frac{E+m}{(2E)^2}\begin{pmatrix}\Sigma\dot{\Sigma}-\dot{\Sigma}\Sigma & \dot{\Sigma}+\Sigma\dot{\Sigma}\Sigma\\ \dot{\Sigma}+\Sigma\dot{\Sigma}\Sigma & \dot{\Sigma}\Sigma-\Sigma\dot{\Sigma}\end{pmatrix}\right]\\
&= \frac{1}{2m}\left[\begin{pmatrix}E-m & -\boldsymbol{\sigma}\cdot\mathbf{p}\\ \boldsymbol{\sigma}\cdot\mathbf{p} & -E-m\end{pmatrix} - i\hbar\frac{1}{(2E)^2}\begin{pmatrix}2i\boldsymbol{\sigma}\cdot\mathbf{p}\times\dot{\mathbf{p}} & 2m\boldsymbol{\sigma}\cdot\dot{\mathbf{p}}\\ 2m\boldsymbol{\sigma}\cdot\dot{\mathbf{p}} & -2i\boldsymbol{\sigma}\cdot\mathbf{p}\times\dot{\mathbf{p}}\end{pmatrix}\right],
\end{aligned}
$$

leading to

$$
v_\alpha(p)\bar{v}^\alpha(p) = \frac{\gamma\cdot p - m}{2m} - \frac{\hbar}{m(2E)^2}\gamma^5\boldsymbol{\gamma}\cdot\mathbf{p}\times\dot{\mathbf{p}} - \frac{i\hbar}{(2E)^2}\gamma^0\boldsymbol{\gamma}\cdot\dot{\mathbf{p}}\,. \tag{793}
$$

5. Split time spinor identities

The following identities are for when the time-dependent momenta are evaluated at different times. The identities are the expansion of the zeroth

order spinors under the transformation $(t,t') \to (\bar{t},\eta)$ to order $\hbar$, where

$$t = \bar{t} - \frac{\hbar}{2}\eta\,, \tag{794}$$

$$t' = \bar{t} + \frac{\hbar}{2}\eta\,. \tag{795}$$

All momenta are p and thus we drop the subscripts. Un-primed terms are evaluated at t, primed terms evaluated at t' and barred terms evaluated at $\bar{t}$. Using this transformation, we have the following:

$$U^\dagger(t)U(t') \to I - \frac{i\hbar\eta}{2\bar{E}}\bar{U}^\dagger(\bar{t})\Lambda(\bar{t})U(\bar{t}) + \mathcal{O}(\hbar^2) \tag{796}$$

$$\Sigma(t)\Sigma(t') \to \frac{\bar{\mathbf{p}}^2}{\bar{E}+m} + i\hbar\eta\bar{\Lambda}\,, \tag{797}$$

where the energy-momenta on the right hand side are evaluated at $\bar{t}$.

5.1. Particle split time spinors.

5.1.1.

$$\begin{aligned}
\bar{u}^{(0)}_\alpha(p,t)u^{(0)}_\alpha(p,t') &= \frac{\sqrt{(E+m)(E'+m)}}{2m}\begin{pmatrix} s^\dagger_\alpha U^\dagger & -s^\dagger_\alpha U^\dagger\Sigma \end{pmatrix}\begin{pmatrix} U's_\alpha \\ \Sigma'U's_\alpha \end{pmatrix} \\
&= \frac{\bar{E}+m}{2m}s^\dagger_\alpha U^\dagger\left(1-\Sigma\Sigma'\right)U's_\alpha \\
&= \frac{\bar{E}+m}{2m}s^\dagger_\alpha U^\dagger\left(\frac{2m}{\bar{E}+m} - \frac{i\hbar\eta\bar{\Lambda}}{\bar{E}+m}\right)U's_\alpha \\
&= s^\dagger_\alpha U^\dagger U's_\alpha - \frac{i\hbar\eta}{2m}s^\dagger_\alpha\bar{U}^\dagger\bar{\Lambda}\bar{U}s_\alpha \\
&= s^\dagger_\alpha s_\alpha - \frac{i\hbar\eta}{2\bar{E}}s^\dagger_\alpha\bar{U}^\dagger\bar{\Lambda}\bar{U}s_\alpha - \frac{i\hbar\eta}{2m}s^\dagger_\alpha\bar{U}^\dagger\bar{\Lambda}\bar{U} \\
&= 1 - \frac{i\hbar\eta}{2m}\frac{\bar{\boldsymbol{\xi}}\cdot\bar{\mathbf{p}}\times\dot{\bar{\mathbf{p}}}}{\bar{E}}\,.
\end{aligned} \tag{798}$$

5.1.2.

$$
\begin{aligned}
\bar{u}^{(0)}_\alpha(p,t)\gamma^0 u^{(0)}_\alpha(p,t') &= \frac{\sqrt{(E+m)(E'+m)}}{2m}\left(s^\dagger_\alpha U^\dagger \quad s^\dagger_\alpha U^\dagger \Sigma\right)\begin{pmatrix} U' s_\alpha \\ \Sigma' U' s_\alpha \end{pmatrix} \\
&= \frac{\bar{E}+m}{2m} s^\dagger_\alpha U^\dagger \left(1+\Sigma\Sigma'\right) U' s_\alpha \\
&= \frac{\bar{E}+m}{2m} s^\dagger_\alpha U^\dagger \left(\frac{2\bar{E}}{\bar{E}+m} + \frac{i\hbar\eta\bar{\Lambda}}{\bar{E}+m}\right) U' s_\alpha \\
&= \frac{\bar{E}}{m}\left(1 - \frac{i\hbar\eta}{2\bar{E}} s^\dagger_\alpha \bar{U}^\dagger \bar{\Lambda}\bar{U} s_\alpha\right) + \frac{i\hbar\eta}{2m} s^\dagger_\alpha \bar{U}^\dagger \bar{\Lambda}\bar{U} \\
&= \frac{\bar{E}}{m}\,.
\end{aligned}
\tag{799}
$$

5.1.3.

$$
\begin{aligned}
&\bar{u}^{(0)}_\alpha(p,t)\boldsymbol{\gamma} u^{(0)}_\alpha(p,t') \\
&= \frac{\sqrt{(E+m)(E'+m)}}{2m}\left(s^\dagger_\alpha U^\dagger \Sigma\boldsymbol{\sigma} \quad s^\dagger_\alpha U^\dagger \boldsymbol{\sigma}\right)\begin{pmatrix} U' s_\alpha \\ \Sigma' U' s_\alpha \end{pmatrix} \\
&= \frac{\bar{E}+m}{2m} s^\dagger_\alpha U^\dagger \left(\Sigma\boldsymbol{\sigma} + \boldsymbol{\sigma}\Sigma'\right) U' s_\alpha \\
&= \frac{\bar{E}+m}{2m} s^\dagger_\alpha U^\dagger \left(\left(\bar{\Sigma} - \frac{\hbar\eta}{2}\dot{\bar{\Sigma}}\right)\boldsymbol{\sigma} + \boldsymbol{\sigma}\left(\bar{\Sigma} + \frac{\hbar\eta}{2}\dot{\bar{\Sigma}}\right)\right) U' s_\alpha \\
&= \frac{\bar{E}+m}{2m} s^\dagger_\alpha U^\dagger \left(\bar{\Sigma}\boldsymbol{\sigma} + \boldsymbol{\sigma}\bar{\Sigma} - \frac{\hbar\eta}{2}\left(\dot{\bar{\Sigma}}\boldsymbol{\sigma} - \boldsymbol{\sigma}\dot{\bar{\Sigma}}\right)\right) U' s_\alpha \\
&= \frac{\bar{E}+m}{2m} s^\dagger_\alpha U^\dagger \frac{2\bar{\mathbf{p}}}{\bar{E}+m} U' s_\alpha - \frac{\bar{E}+m}{2m}\frac{\hbar\eta}{2} s^\dagger_\alpha \bar{U}^\dagger \frac{2i}{\bar{E}+m}\boldsymbol{\sigma}\times\left(\dot{\bar{\mathbf{p}}} - \frac{\dot{\bar{E}}\bar{\mathbf{p}}}{\bar{E}+m}\right)\bar{U} s_\alpha \\
&= \frac{\bar{\mathbf{p}}}{m}\left(1 - \frac{i\hbar\eta}{2\bar{E}} s^\dagger \bar{U}^\dagger \bar{\Lambda}\bar{U} s_\alpha\right) - \frac{i\hbar\eta}{2m} s^\dagger \bar{U}^\dagger \boldsymbol{\sigma}\times\left(\dot{\bar{\mathbf{p}}} - \frac{\dot{\bar{E}}\bar{\mathbf{p}}}{\bar{E}+m}\right)\bar{U} s_\alpha \\
&= \frac{\bar{\mathbf{p}}}{m} - \frac{i\hbar\eta}{2m}\left(\frac{\bar{\boldsymbol{\xi}}\cdot\bar{\mathbf{p}}\times\dot{\bar{\mathbf{p}}}\bar{\mathbf{p}}}{\bar{E}\left(\bar{E}+m\right)} + \bar{\boldsymbol{\xi}}\times\left(\dot{\bar{\mathbf{p}}} - \frac{\dot{\bar{E}}\bar{\mathbf{p}}}{\bar{E}+m}\right)\right)\,.
\end{aligned}
\tag{800}
$$

5.1.4. *Inner spinors.*

$$
\begin{aligned}
&u_\alpha^{(0)}(p,t)\bar{u}^{(0)\alpha}(p,t') \\
&= \sum_\alpha \frac{\sqrt{(E+m)(E'+m)}}{2m}\left[\begin{pmatrix} Us_\alpha \\ \Sigma U s_\alpha \end{pmatrix}\begin{pmatrix} s_\alpha^\dagger U^{\dagger\prime} & -s_\alpha^\dagger U^{\dagger\prime}\Sigma' \end{pmatrix}\right] \\
&= \sum_\alpha \frac{\bar{E}+m}{2m}\begin{pmatrix} Us_\alpha s_\alpha^\dagger U^{\dagger\prime} & -Us_\alpha s_\alpha^\dagger U^{\dagger\prime}\Sigma' \\ \Sigma U s_\alpha s_\alpha^\dagger U^{\dagger\prime} & -\Sigma U s_\alpha s_\alpha^\dagger U^{\dagger\prime}\Sigma' \end{pmatrix} \\
&= \frac{\bar{E}+m}{2m}\begin{pmatrix} UU^{\dagger\prime} & -UU^{\dagger\prime}\Sigma' \\ \Sigma UU^{\dagger\prime} & -\Sigma UU^{\dagger\prime}\Sigma' \end{pmatrix}.
\end{aligned}
$$

Expanding the elements in terms of $\hbar$, starting with the unitary operators we have

$$
\begin{aligned}
u_\alpha^{(0)}(p,t)\bar{u}^{(0)\alpha}(p,t') &= \frac{\bar{E}+m}{2m}\left[\begin{pmatrix} I & -\Sigma' \\ \Sigma & -\Sigma\Sigma' \end{pmatrix} + \frac{i\hbar\eta}{2\bar{E}}\begin{pmatrix} \bar{\Lambda} & -\bar{\Lambda}\bar{\Sigma} \\ \bar{\Sigma}\bar{\Lambda} & -\bar{\Sigma}\bar{\Lambda}\bar{\Sigma} \end{pmatrix}\right] \\
&= \frac{\bar{E}+m}{2m}\left[\begin{pmatrix} I & -\bar{\Sigma} \\ \bar{\Sigma} & -\bar{\Sigma}^2 \end{pmatrix} - \frac{\hbar\eta}{2}\begin{pmatrix} 0 & \dot{\bar{\Sigma}} \\ \dot{\bar{\Sigma}} & \bar{\Sigma}\dot{\bar{\Sigma}} - \dot{\bar{\Sigma}}\bar{\Sigma} \end{pmatrix}\right. \\
&\quad \left. + \frac{i\hbar\eta}{2\bar{E}}\begin{pmatrix} \bar{\Lambda} & 0 \\ 0 & \frac{\bar{\mathbf{p}}^2\bar{\Lambda}}{\left(\bar{E}+m\right)^2} \end{pmatrix} + \frac{i\hbar\eta}{2\bar{E}}\frac{i\bar{\mathbf{p}}\times\left(\bar{\mathbf{p}}\times\dot{\bar{\mathbf{p}}}\right)}{(\bar{E}+m)^2}\cdot\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}\right] \\
&= \frac{\boldsymbol{\gamma}\cdot\bar{\mathbf{p}}+m}{2m} - \frac{\hbar\eta}{2m.2}\gamma^0\boldsymbol{\gamma}\cdot\left(\dot{\bar{\mathbf{p}}} - \frac{\dot{\bar{E}}\bar{\mathbf{p}}}{\bar{E}+m} + \frac{\bar{\mathbf{p}}\bar{\mathbf{p}}\cdot\dot{\bar{\mathbf{p}}} - \dot{\bar{\mathbf{p}}}\bar{\mathbf{p}}^2}{\bar{E}\left(\bar{E}+m\right)}\right) \\
&\quad + \frac{i\hbar\eta}{2m}\left(\bar{E}+m\right)\begin{pmatrix} \frac{\bar{\Lambda}}{2\bar{E}} & 0 \\ 0 & \bar{\Lambda}\left(\frac{\bar{\mathbf{p}}^2}{2\bar{E}(\bar{E}+m)^2} - \frac{1}{\bar{E}+m}\right) \end{pmatrix},
\end{aligned}
$$

which finally leads to

$$
u_\alpha^{(0)}(p,t)\bar{u}^{(0)\alpha}(p,t') = \frac{\boldsymbol{\gamma}\cdot\bar{\mathbf{p}}+m}{2m} - \frac{i\hbar\eta}{2\bar{E}2m}\gamma^5\boldsymbol{\gamma}\cdot\bar{\mathbf{p}}\times\dot{\bar{\mathbf{p}}} - \frac{\hbar\eta}{2\bar{E}.2}\gamma^0\boldsymbol{\gamma}\cdot\dot{\bar{\mathbf{p}}}. \quad (801)
$$

5.2. Antiparticle Split time spinors.

5.2.1.

$$
\begin{aligned}
\bar{v}_\alpha^{(0)}(p,t)v_\alpha^{(0)}(p,t') &= \frac{\sqrt{(E+m)(E'+m)}}{2m}\begin{pmatrix} s_\alpha^\dagger U^\dagger \Sigma & -s_\alpha^\dagger U^\dagger \end{pmatrix}\begin{pmatrix} \Sigma' U' s_\alpha \\ U' s_\alpha \end{pmatrix} \\
&= \frac{\bar{E}+m}{2m} s_\alpha^\dagger U^\dagger \left(\Sigma\Sigma' - 1\right) U' s_\alpha \\
&= \frac{\bar{E}+m}{2m} s_\alpha^\dagger U^\dagger \left(-\frac{2m}{\bar{E}+m} + \frac{i\hbar\eta\bar{\Lambda}}{\bar{E}+m}\right) U' s_\alpha \\
&= -s_\alpha^\dagger U^\dagger U' s_\alpha + \frac{i\hbar\eta}{2m} s_\alpha^\dagger \bar{U}^\dagger \bar{\Lambda} \bar{U} s_\alpha \\
&= -s_\alpha^\dagger s_\alpha + \frac{i\hbar\eta}{2\bar{E}} s_\alpha^\dagger \bar{U}^\dagger \bar{\Lambda} \bar{U} s_\alpha + \frac{i\hbar\eta}{2m} s_\alpha^\dagger \bar{U}^\dagger \bar{\Lambda} \bar{U} \\
&= -1 + \frac{i\hbar\eta}{2m} \frac{\bar{\boldsymbol{\xi}} \cdot \bar{\mathbf{p}} \times \dot{\mathbf{p}}}{\bar{E}} \,. \qquad (802)
\end{aligned}
$$

5.2.2.

$$
\begin{aligned}
\bar{v}_\alpha^{(0)}(p,t)\gamma^0 v_\alpha^{(0)}(p,t') &= \frac{\sqrt{(E+m)(E'+m)}}{2m}\begin{pmatrix} s_\alpha^\dagger U^\dagger \Sigma & s_\alpha^\dagger U^\dagger \end{pmatrix}\begin{pmatrix} \Sigma' U' s_\alpha \\ U' s_\alpha \end{pmatrix} \\
&= \frac{\bar{E}+m}{2m} s_\alpha^\dagger U^\dagger \left(\Sigma\Sigma' + 1\right) U' s_\alpha \\
&= \frac{\bar{E}+m}{2m} s_\alpha^\dagger U^\dagger \left(\frac{2\bar{E}}{\bar{E}+m} + \frac{i\hbar\eta\bar{\Lambda}}{\bar{E}+m}\right) U' s_\alpha \\
&= \frac{\bar{E}}{m}\left(1 - \frac{i\hbar\eta}{2\bar{E}} s_\alpha^\dagger \bar{U}^\dagger \bar{\Lambda} \bar{U} s_\alpha\right) + \frac{i\hbar\eta}{2m} s_\alpha^\dagger \bar{U}^\dagger \bar{\Lambda} \bar{U} \\
&= \frac{\bar{E}}{m} \,. \qquad (803)
\end{aligned}
$$

5.2.3.

$$
\begin{aligned}
&\bar{v}^{(0)}_\alpha(p,t)\boldsymbol{\gamma} v^{(0)}_\alpha(p,t') \\
&= \frac{\sqrt{(E+m)(E'+m)}}{2m}\begin{pmatrix} s^\dagger_\alpha U^\dagger & s^\dagger_\alpha U^\dagger \Sigma \boldsymbol{\sigma}\boldsymbol{\sigma}\end{pmatrix}\begin{pmatrix}\Sigma' U' s_\alpha \\ U' s_\alpha\end{pmatrix} \\
&= \frac{\bar{E}+m}{2m} s^\dagger_\alpha U^\dagger \left(\boldsymbol{\sigma}\Sigma' + \Sigma\boldsymbol{\sigma}\right) U' s_\alpha \\
&= \frac{\bar{E}+m}{2m} s^\dagger_\alpha U^\dagger \left(\boldsymbol{\sigma}\left(\bar{\Sigma} + \frac{\hbar\eta}{2}\dot{\bar{\Sigma}}\right) + \left(\bar{\Sigma} - \frac{\hbar\eta}{2}\dot{\bar{\Sigma}}\right)\boldsymbol{\sigma}\right) U' s_\alpha \\
&= \frac{\bar{E}+m}{2m} s^\dagger_\alpha U^\dagger \left(\boldsymbol{\sigma}\bar{\Sigma} + \bar{\Sigma}\boldsymbol{\sigma} + \frac{\hbar\eta}{2}\left(\boldsymbol{\sigma}\dot{\bar{\Sigma}} + \dot{\bar{\Sigma}}\boldsymbol{\sigma}\right)\right) U' s_\alpha \\
&= \frac{\bar{E}+m}{2m} s^\dagger_\alpha U^\dagger \frac{2\bar{\mathbf{p}}}{\bar{E}+m} U' s_\alpha \\
&\quad + \frac{\bar{E}+m}{2m}\frac{\hbar\eta}{2} s^\dagger_\alpha \bar{U}^\dagger \left(\frac{-2i}{\bar{E}+m}\right)\boldsymbol{\sigma}\times\left(\dot{\bar{\mathbf{p}}} - \frac{\dot{\bar{E}}\bar{\mathbf{p}}}{\bar{E}+m}\right)\bar{U} s_\alpha \\
&= \frac{\bar{\mathbf{p}}}{m}\left(1 - \frac{i\hbar\eta}{2\bar{E}} s^\dagger \bar{U}^\dagger \bar{\Lambda}\bar{U} s_\alpha\right) - \frac{i\hbar\eta}{2m} s^\dagger \bar{U}^\dagger \boldsymbol{\sigma}\times\left(\dot{\bar{\mathbf{p}}} - \frac{\dot{\bar{E}}\bar{\mathbf{p}}}{\bar{E}+m}\right)\bar{U} s_\alpha\,.
\end{aligned}
$$

Thus

$$
\bar{v}^{(0)}_\alpha(p,t)\boldsymbol{\gamma} v^{(0)}_\alpha(p,t') = \frac{\bar{\mathbf{p}}}{m} - \frac{i\hbar\eta}{2m}\left(\frac{\bar{\boldsymbol{\xi}}\cdot\bar{\mathbf{p}}\times\dot{\bar{\mathbf{p}}}\bar{\mathbf{p}}}{\bar{E}\left(\bar{E}+m\right)} + \bar{\boldsymbol{\xi}}\times\left(\dot{\bar{\mathbf{p}}} - \frac{\dot{\bar{E}}\bar{\mathbf{p}}}{\bar{E}+m}\right)\right)\,. \tag{804}
$$

5.2.4. *Inner spinors.*

$$
\begin{aligned}
&v^{(0)}_\alpha(p,t)\bar{v}^{(0)\alpha}(p,t') \\
&= \sum_\alpha \frac{\sqrt{(E+m)(E'+m)}}{2m}\left[\begin{pmatrix}\Sigma U s_\alpha \\ U s_\alpha\end{pmatrix}\begin{pmatrix} s^\dagger_\alpha U^{\dagger\prime}\Sigma' & -s^\dagger_\alpha U^{\dagger\prime}\end{pmatrix}\right] \\
&= \sum_\alpha \frac{\bar{E}+m}{2m}\begin{pmatrix}\Sigma U s_\alpha s^\dagger_\alpha U^{\dagger\prime}\Sigma' & -\Sigma U s_\alpha s^\dagger_\alpha U^{\dagger\prime} \\ U s_\alpha s^\dagger_\alpha U^{\dagger\prime}\Sigma' & -U s_\alpha s^\dagger_\alpha U^{\dagger\prime}\end{pmatrix} \\
&= \frac{\bar{E}+m}{2m}\begin{pmatrix}\Sigma U U^{\dagger\prime}\Sigma' & -\Sigma U U^{\dagger\prime} \\ U U^{\dagger\prime}\Sigma' & -U U^{\dagger\prime}\end{pmatrix}\,.
\end{aligned}
$$

Expanding the unitary matrices to order $\hbar$,

$$
\begin{aligned}
&v^{(0)}_{\alpha}(p,t)\bar{v}^{(0)\alpha}(p,t') \\
&= \frac{\bar{E}+m}{2m}\left[\begin{pmatrix} \Sigma\Sigma' & -\Sigma \\ \Sigma' & -I \end{pmatrix} + \frac{i\hbar\eta}{2\bar{E}}\begin{pmatrix} \bar{\Sigma}\bar{\Lambda}\bar{\Sigma} & -\bar{\Sigma}\bar{\Lambda} \\ \bar{\Lambda}\bar{\Sigma} & -\bar{\Lambda} \end{pmatrix}\right] \\
&= \frac{\bar{E}+m}{2m}\left[\begin{pmatrix} \bar{\Sigma}^2 & -\bar{\Sigma} \\ \bar{\Sigma} & -I \end{pmatrix} + \frac{\hbar\eta}{2}\begin{pmatrix} \bar{\Sigma}\dot{\bar{\Sigma}} - \dot{\bar{\Sigma}}\bar{\Sigma} & \dot{\bar{\Sigma}} \\ \dot{\bar{\Sigma}} & 0 \end{pmatrix}\right. \\
&\quad \left. + \frac{i\hbar\eta}{2\bar{E}}\begin{pmatrix} -\frac{\bar{\mathbf{p}}^2\bar{\Lambda}}{\left(\bar{E}+m\right)^2} & 0 \\ 0 & -\bar{\Lambda} \end{pmatrix} - \frac{i\hbar\eta}{2\bar{E}}\frac{i\bar{\mathbf{p}}\times\left(\bar{\mathbf{p}}\times\dot{\bar{\mathbf{p}}}\right)}{(\bar{E}+m)^2}\cdot\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}\right] \\
&= \frac{\boldsymbol{\gamma}\cdot\bar{\mathbf{p}} - m}{2m} + \frac{\hbar\eta}{2m.2}\gamma^0\boldsymbol{\gamma}\cdot\left(\dot{\bar{\mathbf{p}}} - \frac{\dot{\bar{E}}\bar{\mathbf{p}}}{\bar{E}+m} + \frac{\bar{\mathbf{p}}\bar{\mathbf{p}}\cdot\dot{\bar{\mathbf{p}}} - \dot{\bar{\mathbf{p}}}\bar{\mathbf{p}}^2}{\bar{E}\left(\bar{E}+m\right)}\right) \\
&\quad + \frac{i\hbar\eta}{2m}\left(\bar{E}+m\right)\begin{pmatrix} \bar{\Lambda}\left(\frac{1}{\bar{E}+m} - \frac{\bar{\mathbf{p}}^2}{2\bar{E}(\bar{E}+m)^2}\right) & 0 \\ 0 & -\frac{\bar{\Lambda}}{2\bar{E}} \end{pmatrix}.
\end{aligned}
$$

Therefore, we have

$$
v^{(0)}_{\alpha}(p,t)\bar{v}^{(0)\alpha}(p,t') = \frac{\boldsymbol{\gamma}\cdot\bar{\mathbf{p}} + m}{2m} - \frac{i\hbar\eta}{2\bar{E}2m}\gamma^5\boldsymbol{\gamma}\cdot\bar{\mathbf{p}}\times\dot{\bar{\mathbf{p}}} + \frac{\hbar\eta}{2\bar{E}.2}\gamma^0\boldsymbol{\gamma}\cdot\dot{\bar{\mathbf{p}}}\,. \quad (805)
$$

6. Summary of semiclassical spinor identities

In this section, for ease of practical use, we collect together the spinor identities derived in the previous sections. All terms on the right hand side are evaluated at time $\bar{t}$ with energy-momenta $\tilde{p} = (E_p, \tilde{\mathbf{p}})$ and $\tilde{p}_+ = (E_{p_+}, \tilde{\mathbf{p}}_+)$ for the particle and antiparticle identities respectively. These results are quoted in the main body of this work, when needed for the evaluation of the spinor combinations found in the forward scattering loops.

$$\bar{u}_\alpha(p,t)u_\alpha(p,t') = 1 - \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}}{E_p^2} - \frac{i\hbar\eta}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\mathbf{p}}}{E_p} + \mathcal{O}(\hbar^2)\,, \tag{806}$$

$$\bar{u}_\alpha(p,t)\gamma^0 u_\alpha(p,t') = \frac{E_p}{m} + \mathcal{O}(\hbar^2)\,, \tag{807}$$

$$\begin{aligned}\bar{u}_\alpha(p,t)\boldsymbol{\gamma}u_\alpha(p,t') = \frac{\tilde{\mathbf{p}}}{m} &- \frac{\hbar}{2mE_p}\left(\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\,\tilde{\mathbf{p}}}{E_p(E_p+m)} + \boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}} - \frac{\dot{E}_p\tilde{\mathbf{p}}}{E_p+m}\right)\right)\\ &- \frac{i\hbar\eta}{2m}\left(\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}}\,\tilde{\mathbf{p}}}{E_p(E_p+m)} + \boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}} - \frac{\dot{E}_p\tilde{\mathbf{p}}}{E_p+m}\right)\right) + \mathcal{O}(\hbar^2)\,,\end{aligned} \tag{808}$$

$$\begin{aligned}u_\alpha(p,t)\bar{u}^\alpha(p,t') = \frac{\gamma\cdot\tilde{p}+m}{2m} &+ \frac{\hbar}{m(2E_p)^2}\gamma^5\boldsymbol{\gamma}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}} - \frac{i\hbar}{(2E_p)^2}\gamma^0\boldsymbol{\gamma}\cdot\dot{\tilde{\mathbf{p}}}\\ &- \frac{i\hbar\eta}{2m2E_p}\gamma^5\boldsymbol{\gamma}\cdot\tilde{\mathbf{p}}\times\dot{\tilde{\mathbf{p}}} - \frac{\hbar\eta}{2E_p.2}\gamma^0\boldsymbol{\gamma}\cdot\dot{\tilde{\mathbf{p}}} + \mathcal{O}(\hbar^2)\,.\end{aligned} \tag{809}$$

$$\bar{v}_\alpha(p,t)v_\alpha(p,t') = -1 - \frac{\hbar}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}_+\times\dot{\tilde{\mathbf{p}}}_+}{E_{p_+}^2} + \frac{i\hbar\eta}{2m}\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}_+\times\dot{\mathbf{p}}}{E_{p_+}} + \mathcal{O}(\hbar^2)\,, \tag{810}$$

$$\bar{v}_\alpha(p,t)\gamma^0 v_\alpha(p,t') = \frac{E_{p_+}}{m} + \mathcal{O}(\hbar^2)\,, \tag{811}$$

$$\begin{aligned}\bar{v}_\alpha(p,t)\boldsymbol{\gamma}v_\alpha(p,t') = \frac{\tilde{\mathbf{p}}_+}{m}&\\ &+ \frac{\hbar}{2mE_{p_+}}\left(\frac{\boldsymbol{\xi}\cdot\mathbf{p}\times\dot{\tilde{\mathbf{p}}}_+\,\tilde{\mathbf{p}}_+}{E_{p_+}(E_{p_+}+m)} + \boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}}_+ - \frac{\dot{E}_{p_+}\tilde{\mathbf{p}}_+}{E_{p_+}+m}\right)\right)\\ &- \frac{i\hbar\eta}{2m}\left(\frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}_+\times\dot{\tilde{\mathbf{p}}}_+\,\tilde{\mathbf{p}}_+}{E_{p_+}(E_{p_+}+m)} + \boldsymbol{\xi}\times\left(\dot{\tilde{\mathbf{p}}}_+ - \frac{\dot{E}_{p_+}\mathbf{p}}{E_{p_+}+m}\right)\right) + \mathcal{O}(\hbar^2)\,,\end{aligned} \tag{812}$$

$$\begin{aligned}v_\alpha(p,t)\bar{v}^\alpha(p,t') = \frac{\gamma\cdot\tilde{p}_+-m}{2m} &- \frac{\hbar}{m(2E_{p_+})^2}\gamma^5\boldsymbol{\gamma}\cdot\tilde{\mathbf{p}}_+\times\dot{\tilde{\mathbf{p}}}_+ - \frac{i\hbar}{(2E_{p_+})^2}\gamma^0\boldsymbol{\gamma}\cdot\dot{\tilde{\mathbf{p}}}_+\\ &- \frac{i\hbar\eta}{2m2E_{p_+}}\gamma^5\boldsymbol{\gamma}\cdot\tilde{\mathbf{p}}_+\times\dot{\tilde{\mathbf{p}}}_+ + \frac{\hbar\eta}{2E_{p_+}.2}\gamma^0\boldsymbol{\gamma}\cdot\dot{\tilde{\mathbf{p}}}_+ + \mathcal{O}(\hbar^2)\,.\end{aligned} \tag{813}$$

The following identity is for equal momenta zeroth order spinors:

$$\bar{u}_\alpha^{(0)}(p)\gamma^5\boldsymbol{\gamma}u_\alpha^{(0)}(p) = -\left(\boldsymbol{\xi} + \frac{\boldsymbol{\xi}\cdot\tilde{\mathbf{p}}\,\tilde{\mathbf{p}}}{m(E+m)}\right)\,. \tag{814}$$

APPENDIX B

Interaction Hamiltonian for the Scalar field

In this appendix we calculate the interaction Hamiltonian for the complex scalar field. The result is in contrast with those cases, such as the spinor field, where the interaction Hamiltonian and Lagrangian are the negative of each other. For simplicity, let us use natural units ($\hbar = 1$) here. Consider the classical Lagrangian density for a charged scalar field interacting with electromagnetic field and in the presence of a background potential V:

$$\mathcal{L} = (\mathcal{D}_\mu \varphi)^\dagger \mathcal{D}^\mu \varphi - m^2 \varphi^\dagger \varphi \,, \tag{815}$$

where $\mathcal{D}_\mu \varphi = (D_\mu + ieA_\mu)\varphi$ and $D_\mu \varphi = (\partial_\mu + iV_\mu)\varphi$. This can be written

$$\mathcal{L} = (\mathcal{D}_0\varphi)^\dagger \mathcal{D}_0\varphi - (D_i\varphi)^\dagger D_i\varphi - m^2\varphi^\dagger\varphi + ie(\varphi^\dagger D_i\varphi - (D_i\varphi)^\dagger \cdot \varphi)A_i - e^2 A_i A_i \varphi^\dagger \varphi \,, \tag{816}$$

where the indices i are summed over $i = 1, 2, 3$. The canonical conjugate momentum densities are

$$\begin{aligned} \pi_\varphi &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = (\mathcal{D}_0\varphi)^\dagger = \dot{\varphi}^\dagger - i(V_0 + eA_0)\varphi^\dagger \\ \pi_{\varphi^\dagger} &= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\dagger} = \mathcal{D}_0\varphi = \dot{\varphi} + i(V_0 + eA_0)\varphi \,. \end{aligned} \tag{817}$$

Hence

$$\begin{aligned} \dot{\varphi} &= \pi_{\varphi^\dagger} - i(V_0 + eA_0)\varphi \\ \dot{\varphi}^\dagger &= \pi_\varphi + i(V_0 + eA_0)\varphi^\dagger \,. \end{aligned} \tag{818}$$

Thus, the Hamiltonian density is

$$\begin{aligned}
\mathcal{H} &= \pi_\varphi \dot{\varphi} + \pi_{\varphi^\dagger}\dot{\varphi} - \mathcal{L} \\
&= \pi_\varphi(\pi_{\varphi^\dagger} - i(V_0 + eA_0)\varphi) + \pi_{\varphi^\dagger}(\pi_\varphi^\dagger + i(V_0 + eA_0)\varphi^\dagger) \\
&\quad - \pi_\varphi \pi_{\varphi^\dagger} + (D_i\varphi)^\dagger D_i\varphi + m^2\varphi^\dagger\varphi - ie(\varphi^\dagger D_i\varphi - (D_i\varphi)^\dagger \cdot \varphi)A_i + e^2 A_i A_i \varphi^\dagger\varphi \\
&= \pi_\varphi \pi_{\varphi^\dagger} + (D_i\varphi)^\dagger D_i\varphi + m^2\varphi^\dagger\varphi \\
&\quad + i(\varphi^\dagger \pi_{\varphi^\dagger} - \varphi\pi_\varphi)(V_0 + eA_0) - ie(\varphi^\dagger D_i\varphi - (D_i\varphi)^\dagger \cdot \varphi) + e^2 A_i A_i \varphi^\dagger\varphi\,.
\end{aligned} \tag{819}$$

Hence we can decompose the Hamiltonian density into free and interacting parts $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$ with

$$\mathcal{H}_0 = \pi_\varphi \pi_{\varphi^\dagger} + (D_i\varphi)^\dagger D_i\varphi + m^2\varphi^\dagger\varphi + i(\varphi^\dagger \pi_{\varphi^\dagger} - \varphi\pi_\varphi)V_0 \tag{820}$$

$$\mathcal{H}_I = ie(\varphi^\dagger \pi_{\varphi^\dagger} - \varphi\pi_\varphi)A_0 - ie(\varphi^\dagger D_i\varphi - (D_i\varphi)^\dagger \cdot \varphi)A_i + e^2 A_i A_i \varphi^\dagger\varphi\,. \tag{821}$$

Hamilton's equations with $H_0 = \int d^3\mathbf{x}\mathcal{H}_0$ read

$$\begin{aligned}
\dot{\varphi} &= \frac{\delta H_0}{\delta \pi_\varphi} = \pi_{\varphi^\dagger} - iV_0\varphi\,, \\
\dot{\pi}_{\varphi^\dagger} &= -\frac{\delta H_0}{\delta \varphi^\dagger} = D_i D_i \varphi - m^2\varphi - iV_0\pi_{\varphi^\dagger}\,,
\end{aligned}$$

and their conjugates. These equations can be rewritten as

$$(D_\mu D^\mu + m^2)\varphi = 0 \tag{822}$$

$$\pi_{\varphi^\dagger} = D_0\varphi\,, \tag{823}$$

and their conjugates, as expected.

In the interaction picture, φ obeys the Hamilton's equations with H_0, so we can let $\pi_\varphi = (D_0\varphi)^\dagger$ and $\pi_{\varphi^\dagger} = D_0\varphi$. Then

$$\mathcal{H}_I = ie(\varphi^\dagger D_\mu\varphi - (D_\mu\varphi)^\dagger \cdot \varphi)A^\mu + e^2 A_i A_i \varphi^\dagger\varphi\,. \tag{824}$$

The naïve interaction Hamiltonian density is

$$\mathcal{H}_I^{\text{naive}} = ie(\varphi^\dagger D_\mu\varphi - (D_\mu\varphi)^\dagger \cdot \varphi)A^\mu - e^2 A_\mu A^\mu \varphi^\dagger\varphi\,. \tag{825}$$

Overall, the difference is $\mathcal{H}_I - \mathcal{H}_I^{\text{naive}} = e^2 A_0 A_0 \varphi^\dagger\varphi$.

APPENDIX C

Reference: Dirac representation matrices

In this appendix we give the matrix representations of the Pauli, alpha, beta and gamma matrices frequently employed in mathematical discussions on spin and in the Dirac equation. We present the Pauli-Dirac or Standard representations here. These are the representations used in the calculations in this work and thus they are repeated here as a reference for the reader.

1. Pauli Matrices

The three Pauli spin matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{826}$$

These matrices can easily be seen to have the following eigenvalues and eigenvectors:

Pauli Matrix	σ_1		σ_2		σ_3	
Eigenvalue	1	-1	1	-1	1	-1
Eigenvector	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -i \\ 1 \end{pmatrix}$	$\begin{pmatrix} i \\ 1 \end{pmatrix}$

2. Alpha, Beta, Gamma Matrices

2.1. Alpha, Beta Matrices. The alpha and beta matrices, frequently used in the non-covariant form of the Dirac equation and its standrad derivation from the assumption that the equation of motion is first order, are given in the standard representation below:

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \ \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} . \tag{827}$$

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{828}$$

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{829}$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{830}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} . \tag{831}$$

2.2. Gamma Matrices. The gamma matrices, from the covariant form of the Dirac equation and the Feynman slash notation, are given in terms of the alpha and beta matrices by

$$\gamma^0 = \beta, \ \gamma^i = \beta\alpha_i \, . \tag{832}$$

In the standard representation, these matrices are therefore

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{833}$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{834}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \tag{835}$$

$$\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{836}$$

$$\tag{837}$$

Finally, the γ^5 matrix is defined by

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 . \tag{838}$$

Thus, we have in this representation

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{839}$$

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